NP-hardness: Motivation

Erickson, Chapter 12

This course so far

- Design patterns for **efficient algorithms**:
  - Divide-and-Conquer
  - Dynamic Programming
  - Greedy Algorithms
  - Maximum Flow
  - etc.

Which problems have efficient algorithms?

Which problems **don't** have efficient algorithms?

Goal

Classify problems according to their inherent complexity

Complexity Classification

(efficient = polynomial-time computable)

efficient algorithms exist	probably no efficient algorithms
shortest path	longest path
minimum cut	maximum cut
2SAT	3SAT
planar 4-colorability	planar 3-colorability
minimum bipartite vertex-cover	minimum vertex-cover
maximum matching	maximum 3d-matching
linear programming	integer linear programming
primality testing	factoring

(Note: Factoring is not known to be NP-hard!)

Why do we care?

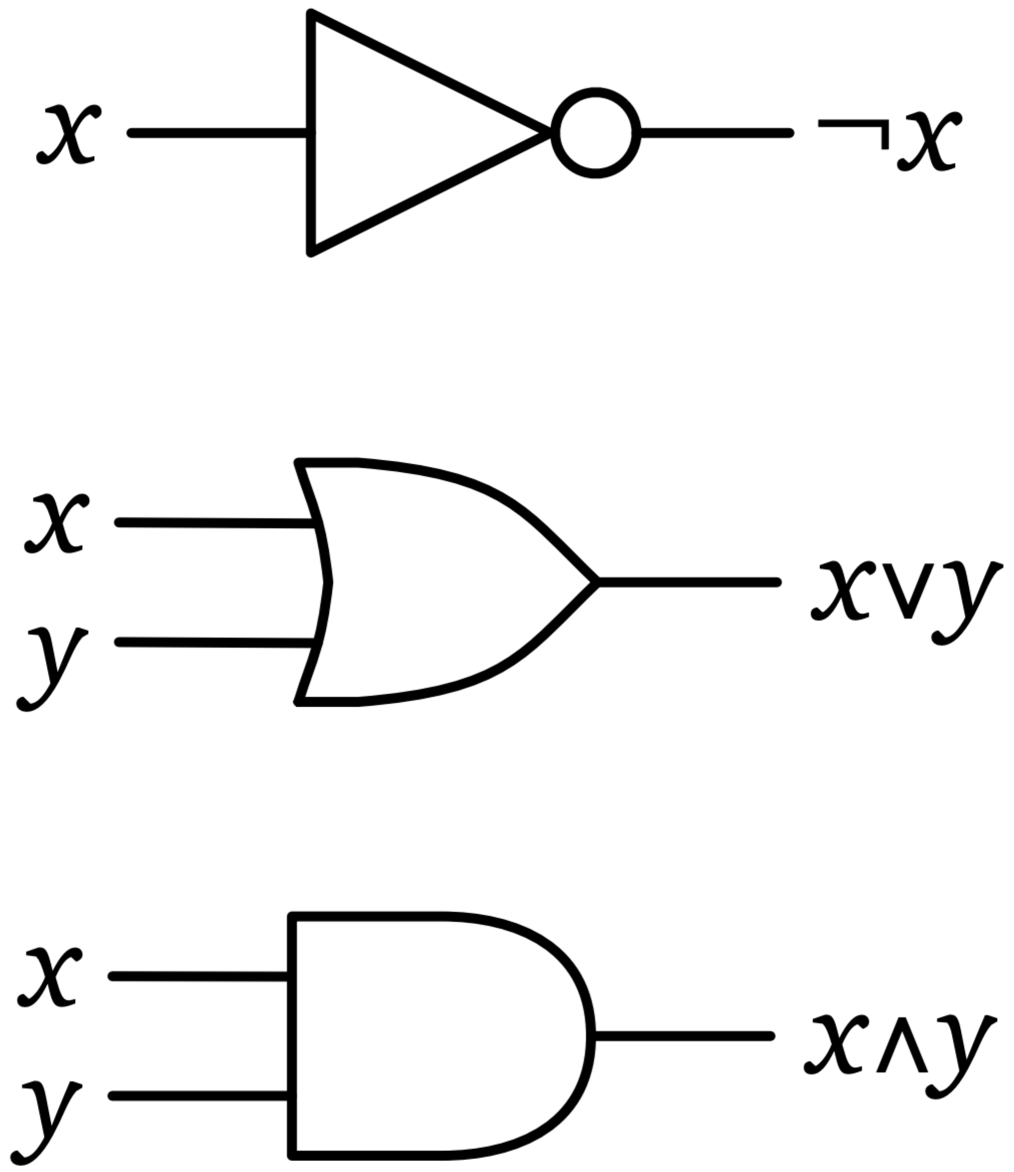
- Know when to change your goals:
  - use heuristics (e.g. SAT-solvers, ILP-solvers, etc.)
  - narrow down your problem
  - use approximation algorithms or fixed-parameter tractable algorithms
- Explain to your employer why neither you nor anyone else can find an efficient algorithm

The Circuit Satisfiability Problem

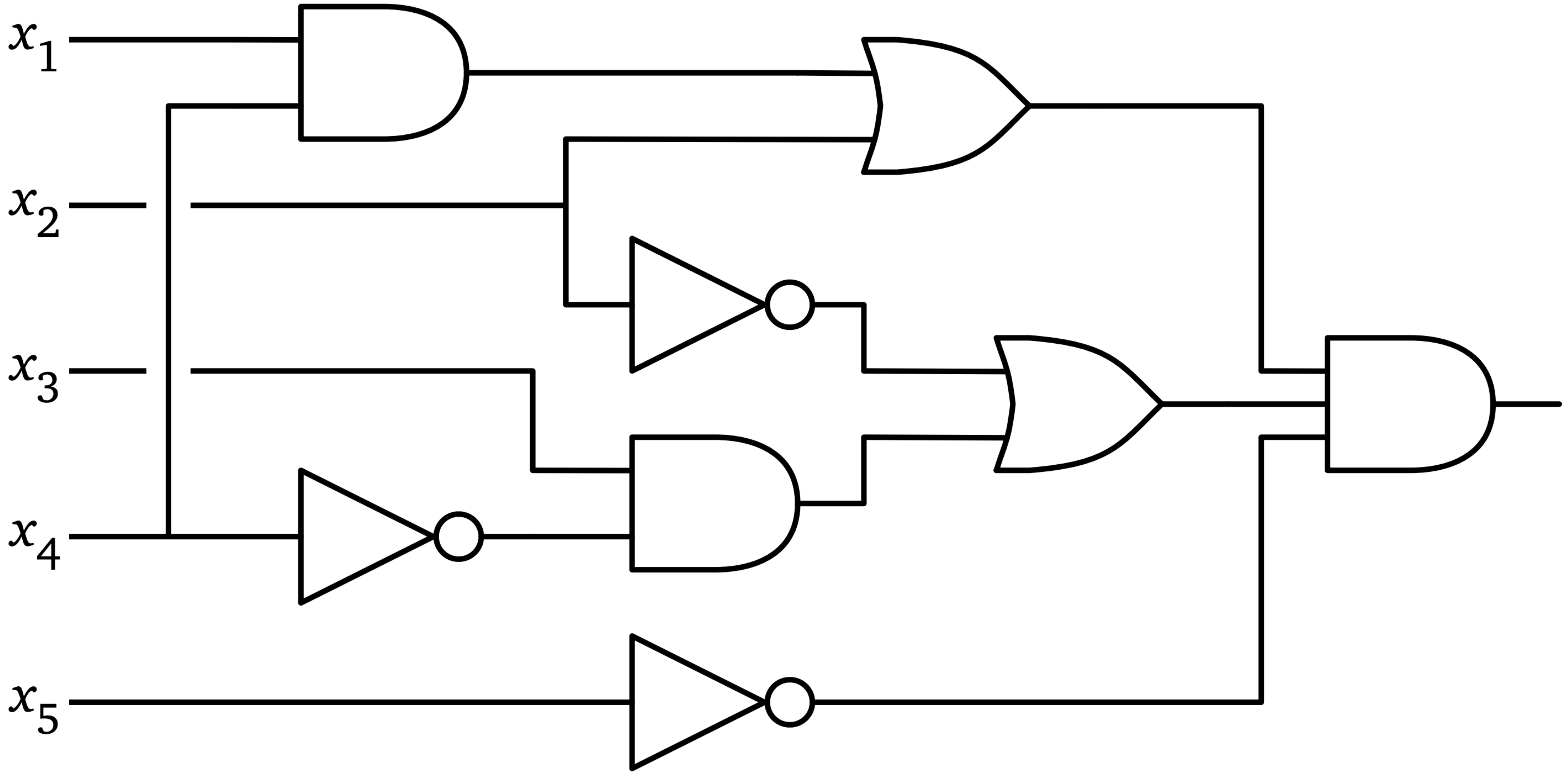
Erickson, Section 12.1

Boolean Circuits

Logical Gates

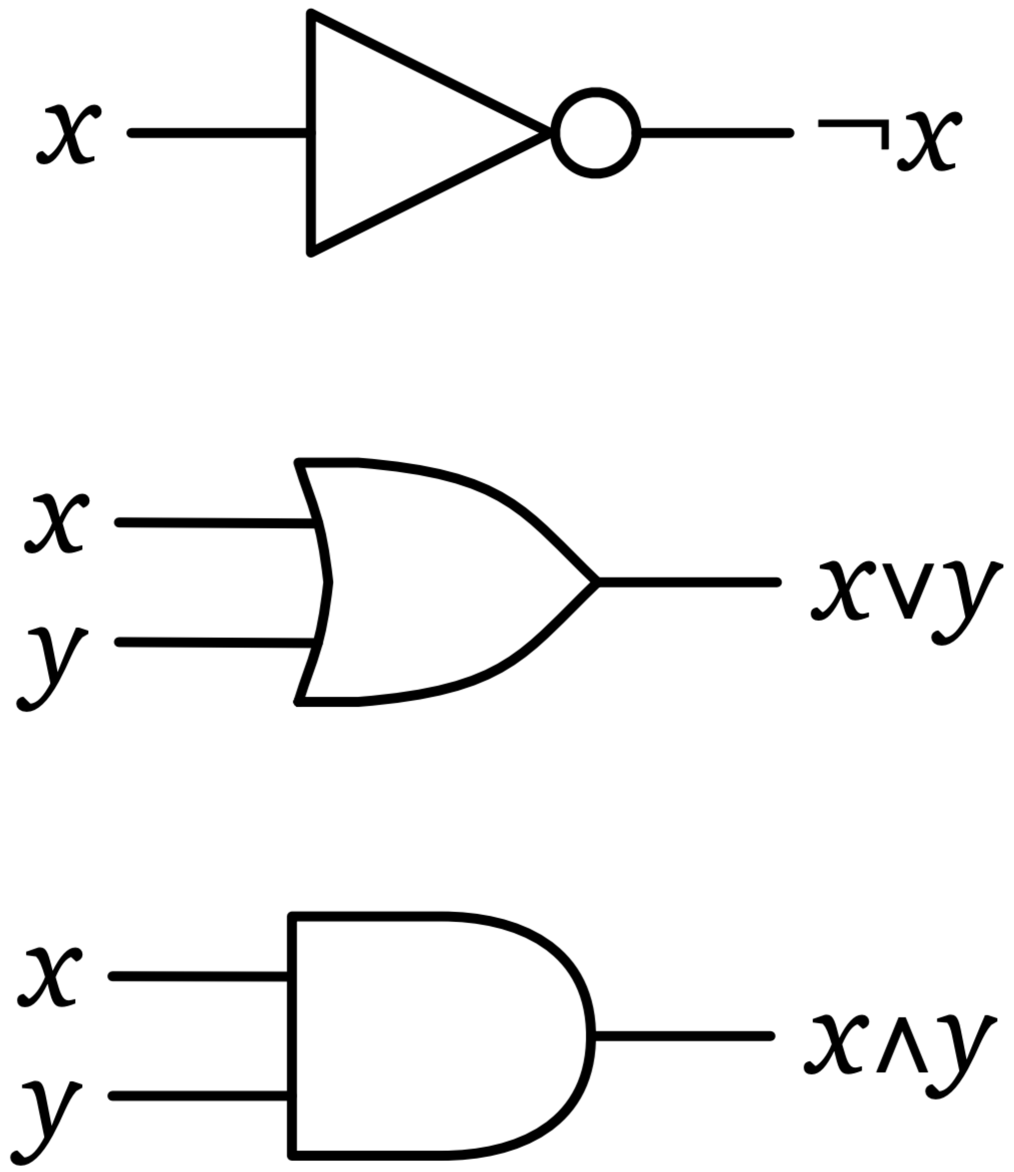


Circuit



Example

Logical Gates



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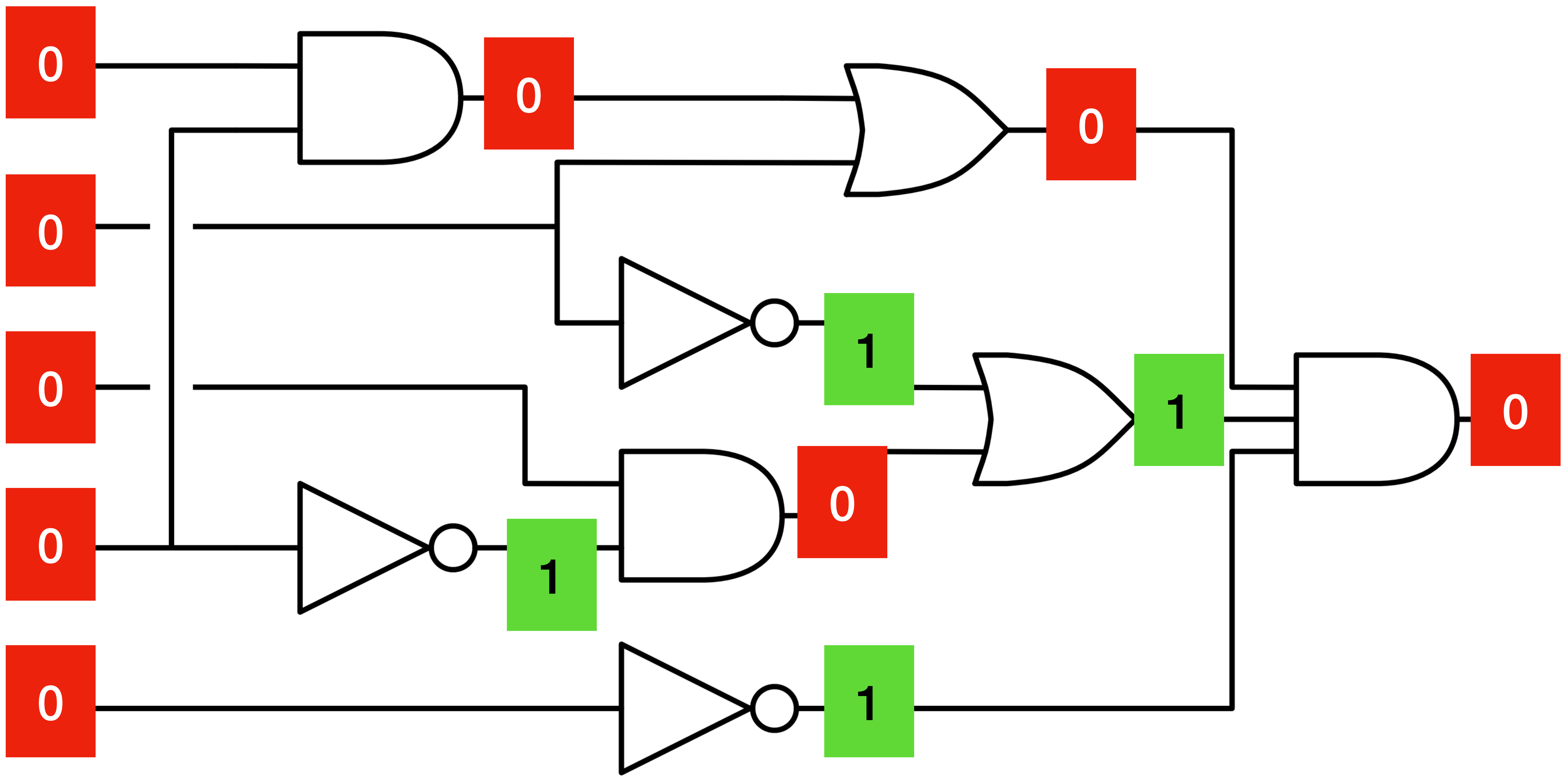
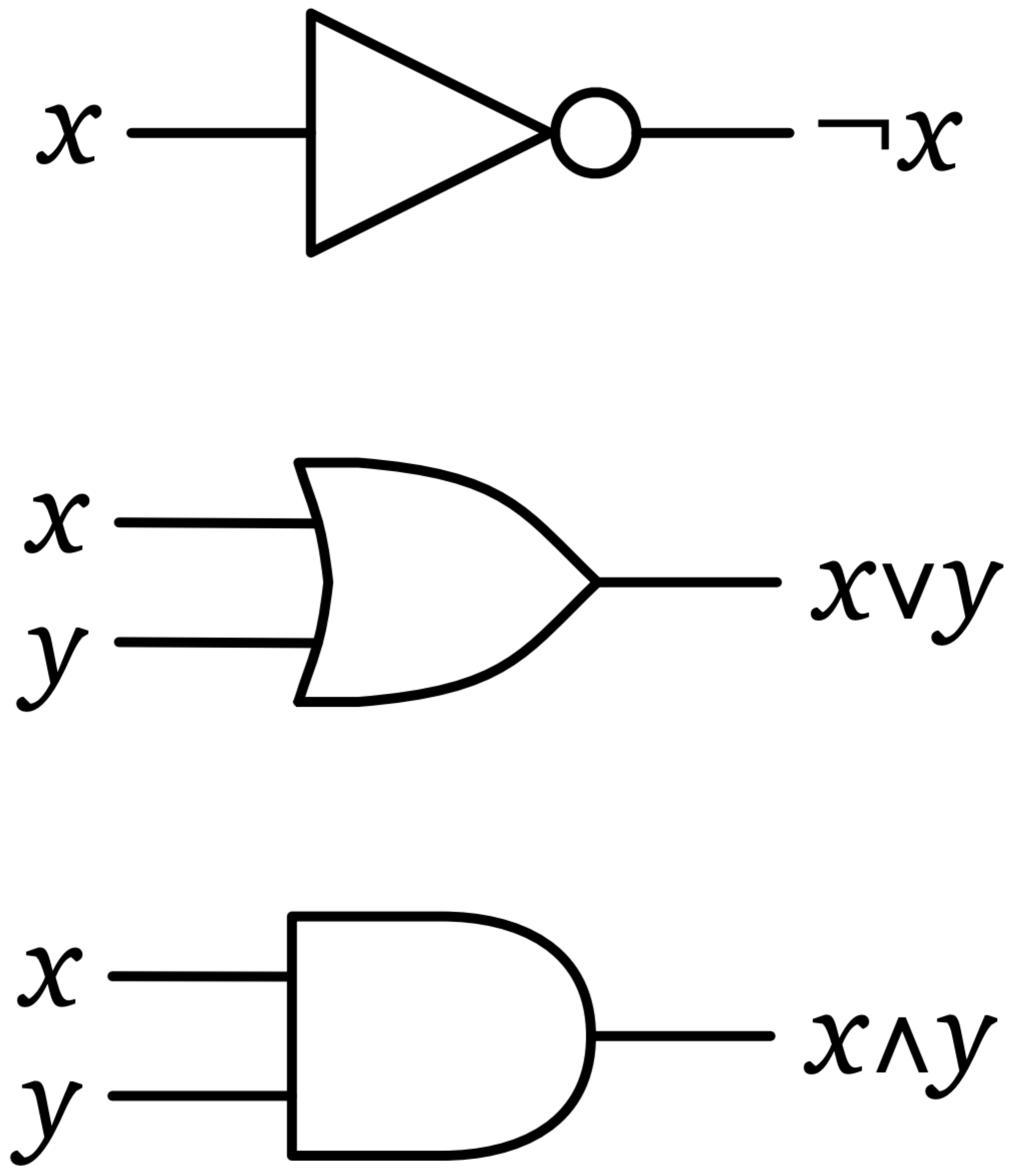


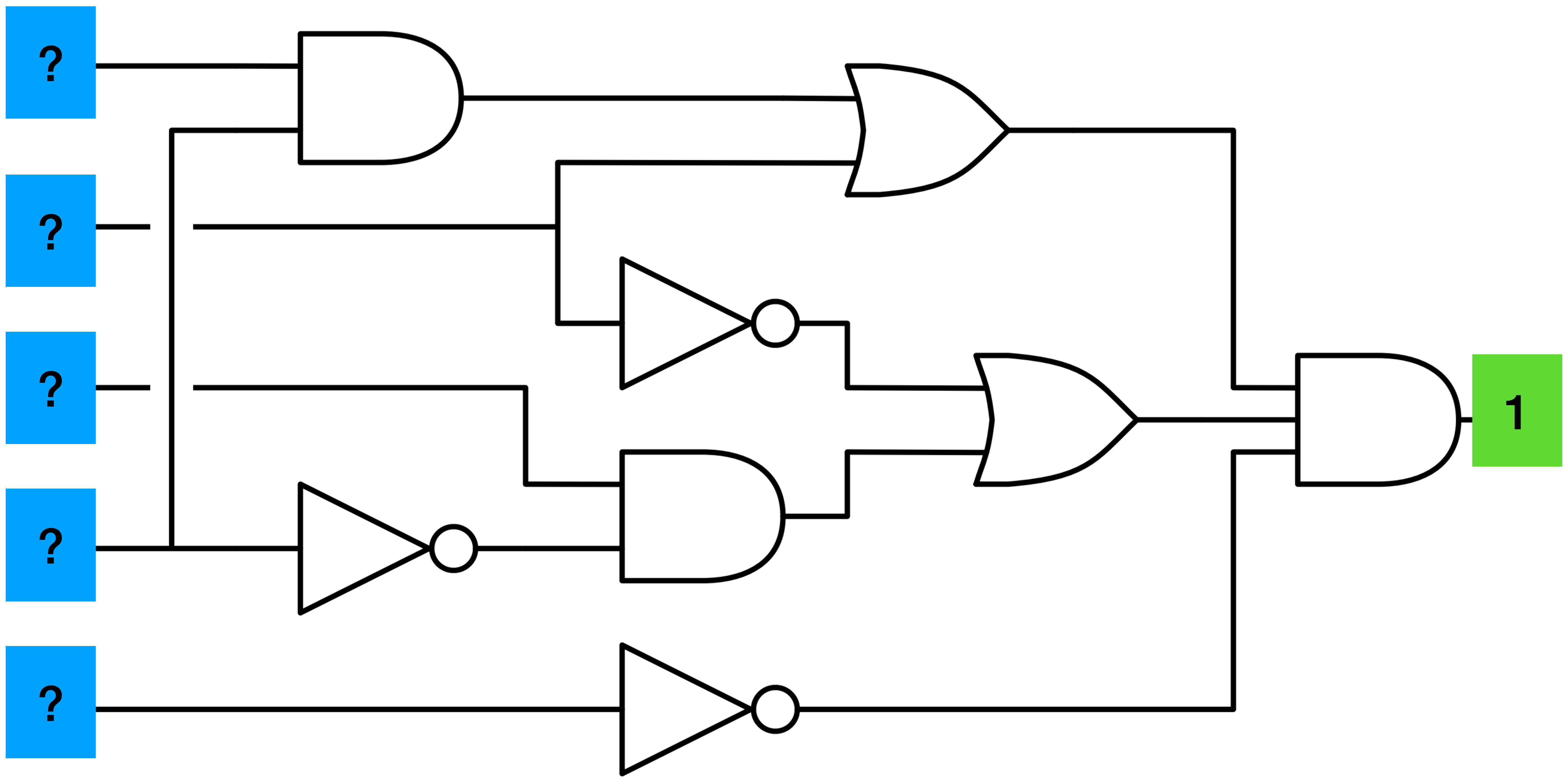
Problem 1: Circuit satisfiability

Is there an assignment so that the circuit outputs 1?

Logical Gates



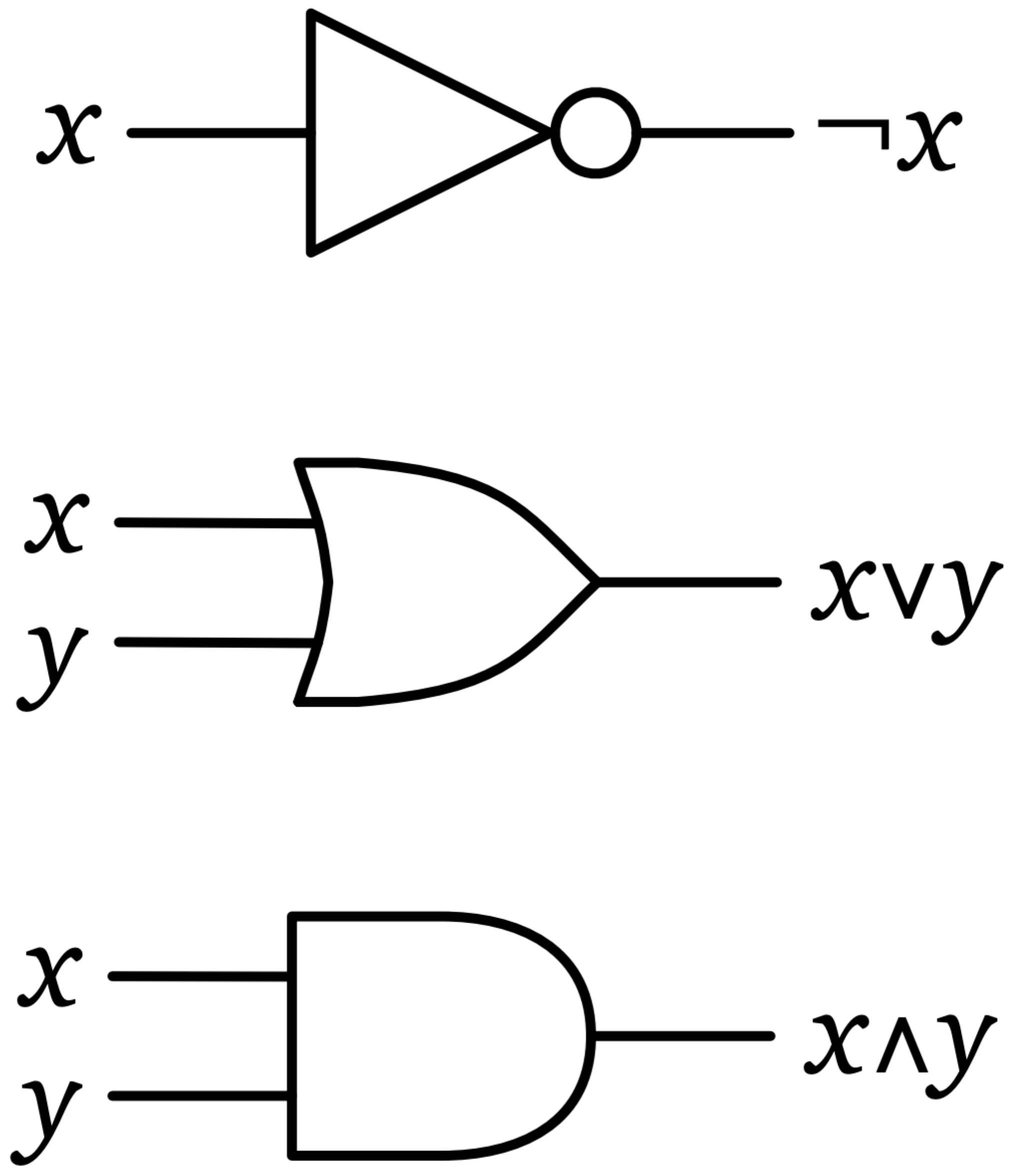
Circuit



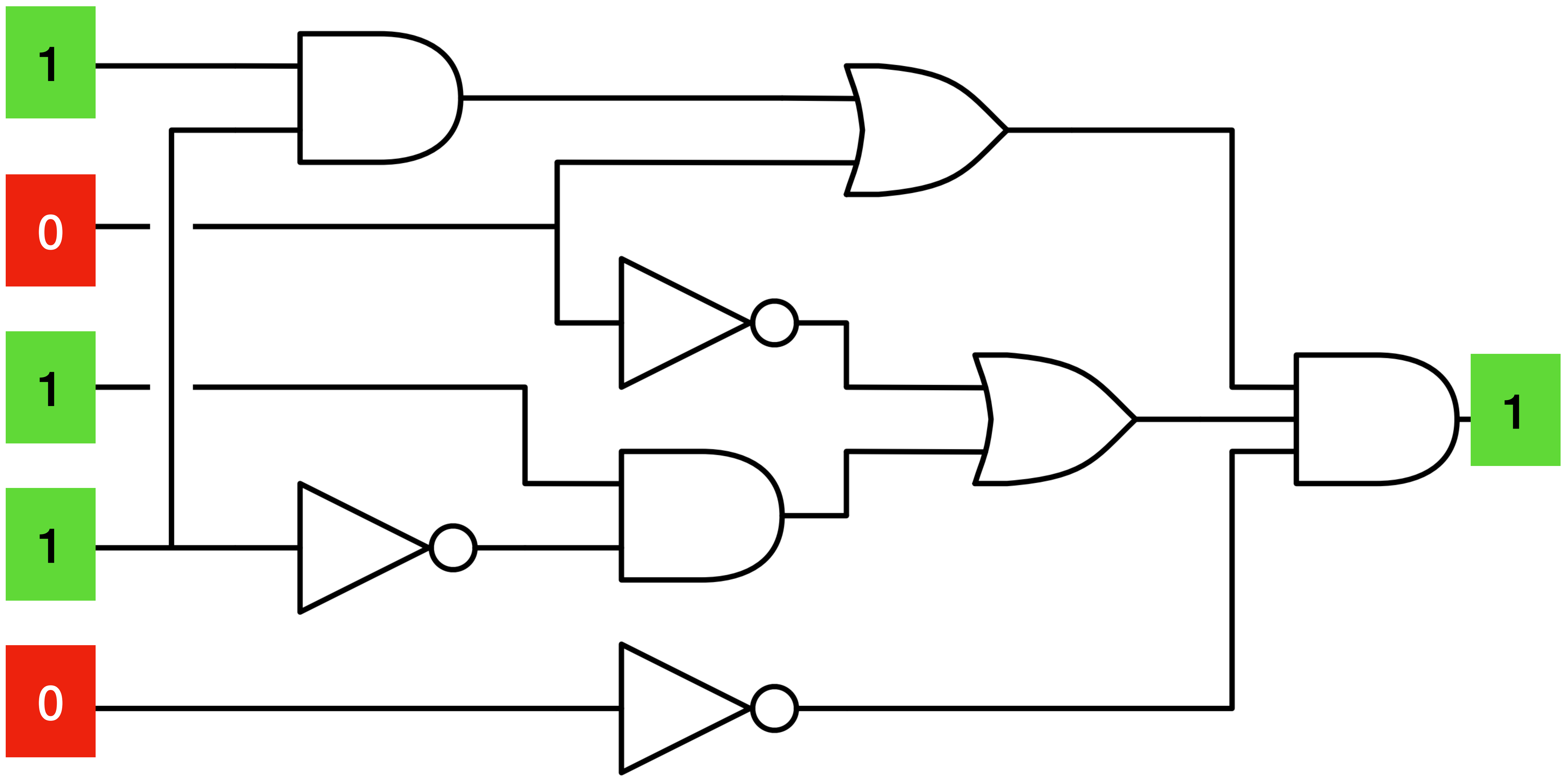
Problem 2: Verification of circuit satisfiability

Verify that the circuit outputs 1 on the given assignment.

Logical Gates



Circuit



This is a **satisfying assignment** of the circuit.

Circuit satisfiability vs verification

- **Definition.** Circuit C is **satisfiable** if it has a satisfying assignment.

- Problem 1: Given circuit C, decide whether C is satisfiable.
- Problem 2: Given circuit C and assignment x, decide whether x satisfies C.

- **Exercise:** Do you think Problem 1 is polynomial-time computable? Do you think Problem 2 polynomial-time computable? Why / Why not?

P versus NP

Erickson, Section 12.2

Decision problems

- **Definition.**
  - finite alphabet Σ, typically $\Sigma = \{0,1\}$
  - decision problem $L \subseteq \Sigma^*$ (also called "language")
- **Example.**
  - CircuitSAT = { Circuit C | C is satisfiable }

Exercise. How do you encode a circuit C as a string in Σ^*?

Algorithm for decision problem

- An algorithm A **solves** L if, for all possible input strings $x \in \Sigma*$, we have:
  - if $x \in L$, then $A(x) = 1$
  - if $x \notin L$, then $A(x) = 0$

- **Example.** An algorithm A solves CircuitSAT if, given any circuit C as input,
  - if C is satisfiable, then A(C)=1
  - if C is not satisfiable, then A(C)=0

Verifier for decision problem

- A **verifier** V for L is an algorithm that is given $x \in \Sigma^*$ as input, such that
  - if $x \in L$, then there exists some $y \in \Sigma^*$ such that $V(x, y) = 1$
  - if $x \notin L$, then, for all $y \in \Sigma^*$, we have $V(x, y) = 0$

- **Exercise.** Write the pseudocode of a polynomial-time verifier V(C, y) for CircuitSAT, that is, an algorithm V that is given a circuit C and an assignment y for C as input.

P versus NP

- **P** = $\left\{ L \subseteq \Sigma^* \mid L \text{ has a polynomial-time algorithm} \right\}$
- **NP** = $\left\{ L \subseteq \Sigma^* \mid L \text{ has a polynomial-time verifier} \right\}$

- **Exercise.** Prove that CircuitSAT is contained in **NP**
- **Open research problem.** Prove that CircuitSAT is not contained in **P**

$$P \overset{?}{\neq} NP$$

NP-hardness, NP-completeness

Erickson, Section 12.3, 12.4

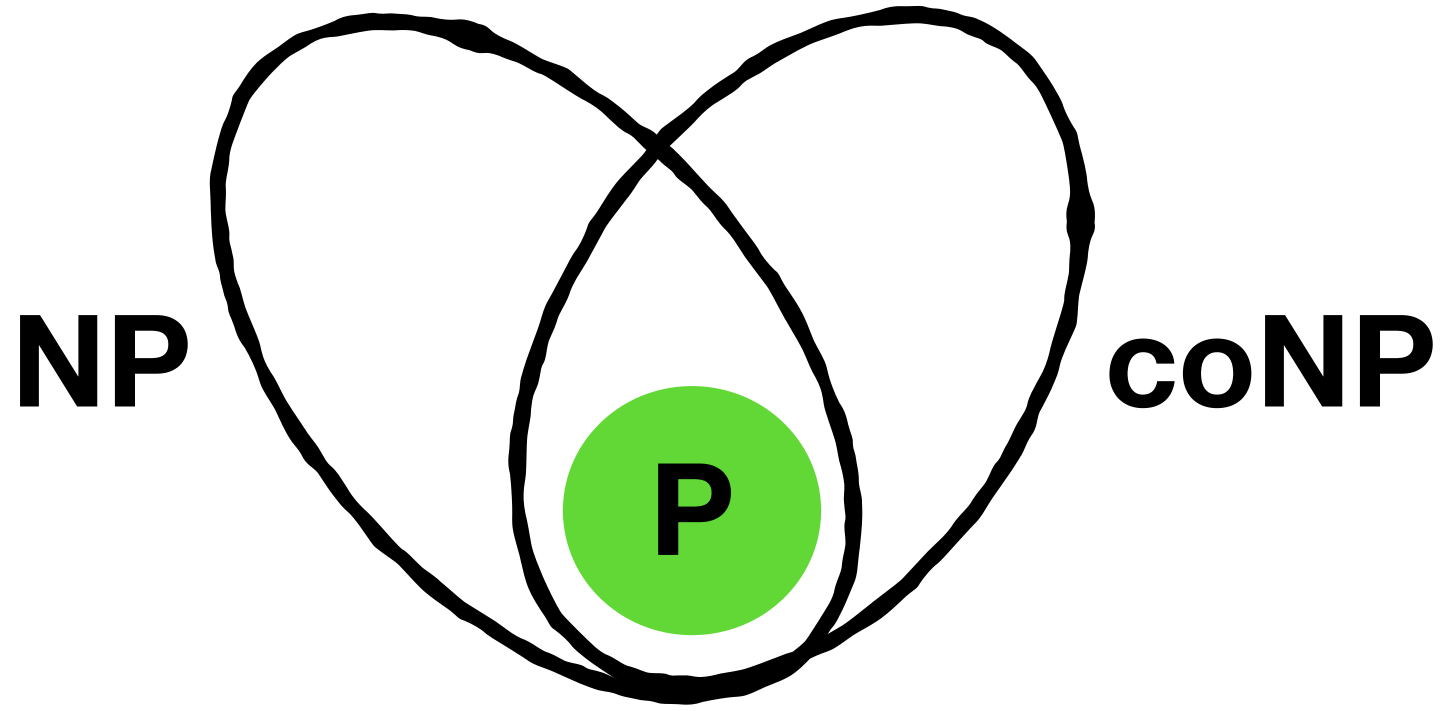
Definition of NP-hardness/NP-completeness

Erickson, Section 12.3

Let $L \subseteq \Sigma^*$ be any decision problem.

- The problem L is **NP-hard** if, for every $L' \in$ **NP**, there is a *polynomial-time reduction* from L' to L.

- The problem L is **NP-complete** if L is **NP**-hard and $L \in$ **NP**



Polynomial-time reduction from L' to L

Erickson, Section 12.4

Suppose we have a magical algorithm **A** that solves L.

Then a **polynomial-time reduction** from L' to L is an algorithm A' that

- takes an input $x' \in \Sigma^*$ for the problem L'
- transforms this input in polynomial time to an input x for the problem L
- executes the magical algorithm **A**(x)
- outputs YES or NO depending on the output of **A**(x).

Cook-Levin Theorem

CircuitSAT is NP-hard

- (We do not prove this theorem here.)

- Here is an important lemma:

If L is **NP**-hard and **P** $\neq$ **NP**, then L $\notin$ **P**.

- **Exercise.** Assuming **P** $\neq$ **NP**, what do you now know about CircuitSAT?

Reductions and SAT

Erickson, Section 12.5

Formula Satisfiability

Exercise.
Which of these formulas is satisfiable?

$\Phi_1 = (x_1 \wedge \overline{x_2}) \vee (x_3 \wedge (x_4 \vee \overline{x_1}))$

$\Phi_2 = (x_1 \wedge \overline{x_2} \wedge (\overline{x_1} \vee x_2))$

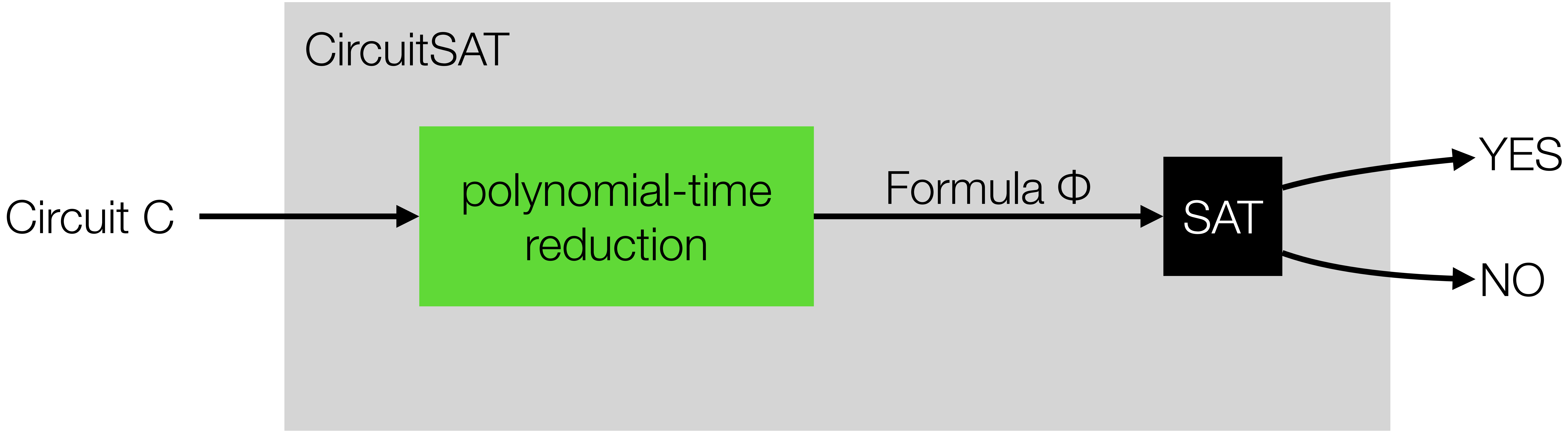
Definition. SAT is the following decision problem:

- Input: Boolean formula Φ
- Question: Is Φ satisfiable?

Is SAT **NP**-hard?

Reductions

- To show that SAT is **NP**-hard,
  we construct a **polynomial-time reduction** from CircuitSAT to SAT:



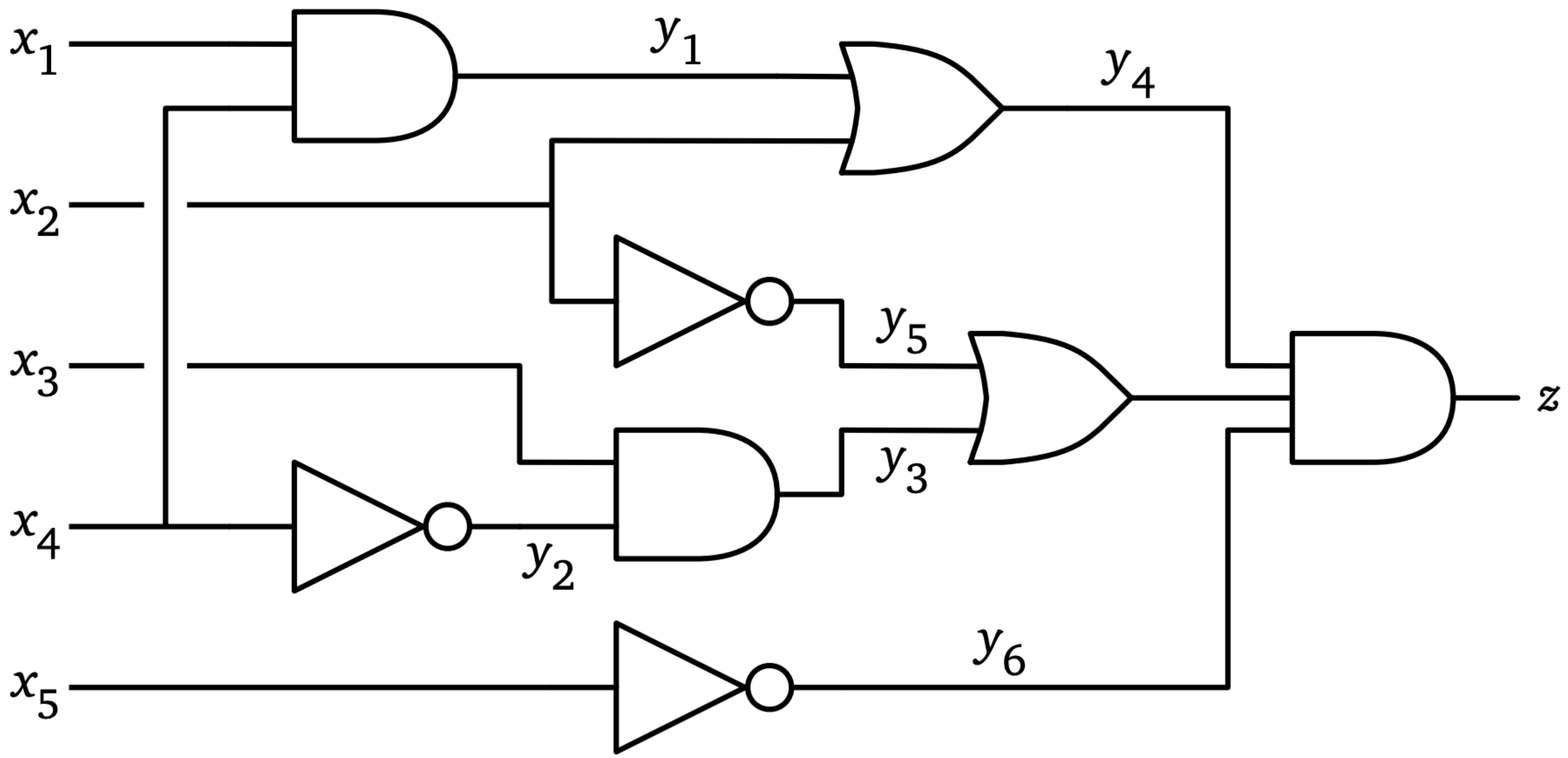
- Any potential algorithm for SAT yields an algorithm for CircuitSAT

Reduction from CircuitSAT to SAT

- **Goal.** Given a circuit C, compute a formula Φ such that C is satisfiable if and only if Φ is satisfiable.
- **Idea.** Introduce a new variable for each wire, then replace each gate with a formula that verifies the computation of the gate.

Reduction from CircuitSAT to SAT

Example



$$(y_1 = x_1 \wedge x_4) \wedge (y_2 = \overline{x_4}) \wedge (y_3 = x_3 \wedge y_2) \wedge (y_4 = y_1 \vee x_2) \wedge$$

$$(y_5 = \overline{x_2}) \wedge (y_6 = \overline{x_5}) \wedge (y_7 = y_3 \vee y_5) \wedge (z = y_4 \wedge y_7 \wedge y_6) \wedge z$$

3SAT is NP-hard

Erickson, Section 12.6

3CNF formulas

Erickson, Section 12.6

- Conjunctive normal form (CNF):

$$(a \vee b \vee c \vee d) \wedge \underbrace{(b \vee \overline{c} \vee \overline{d})}_{\text{clause}} \wedge (\overline{a} \vee c \vee d) \wedge (a \vee \overline{b})$$

- 3CNF formulas: Every clause has width 3.

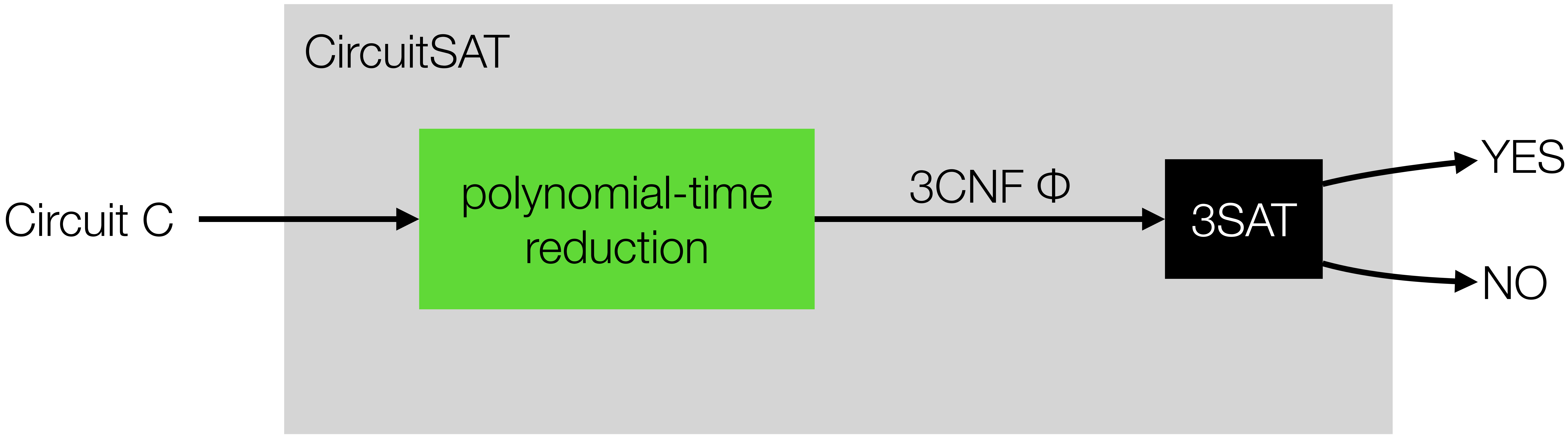
3SAT

- 3SAT is the decision problem:
  - **Input.** 3CNF formula Φ
  - **Question.** Is Φ satisfiable?

- We want to show that 3SAT is NP-hard.

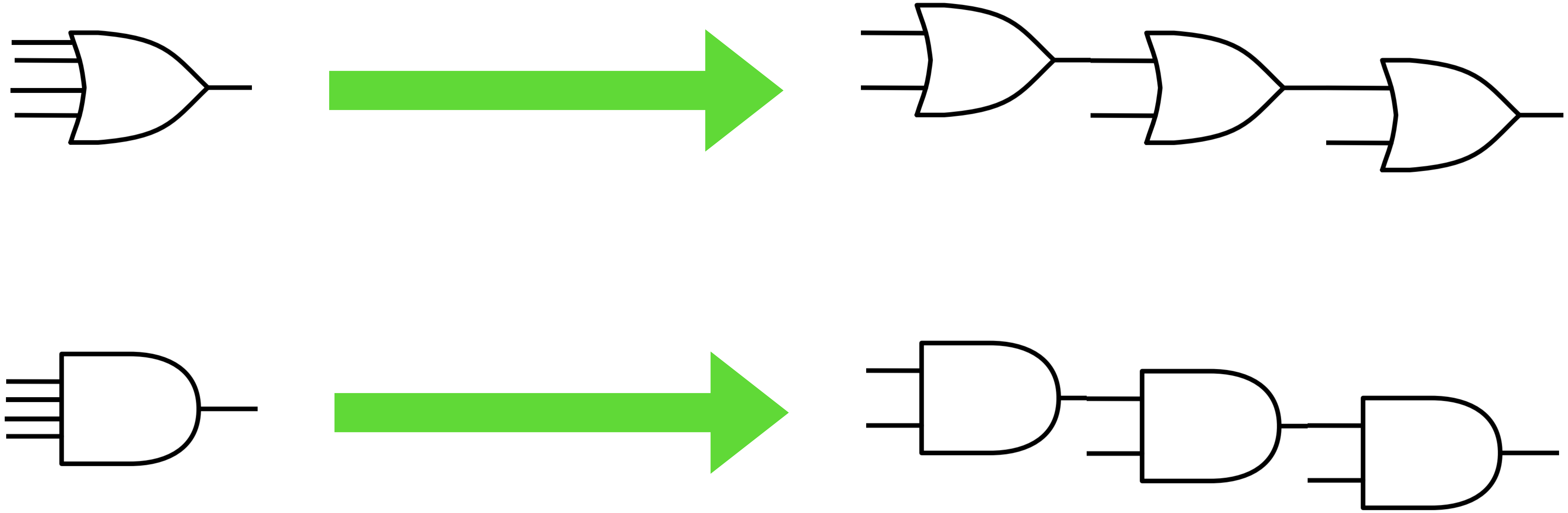
Reduction from CircuitSAT to 3SAT

Overview



Reduction from CircuitSAT to 3SAT

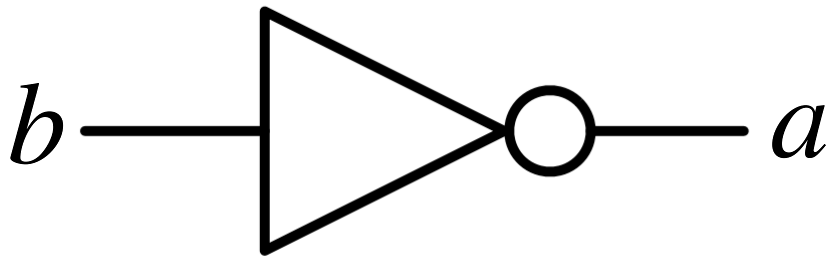
Step 1: Reduce fan-in



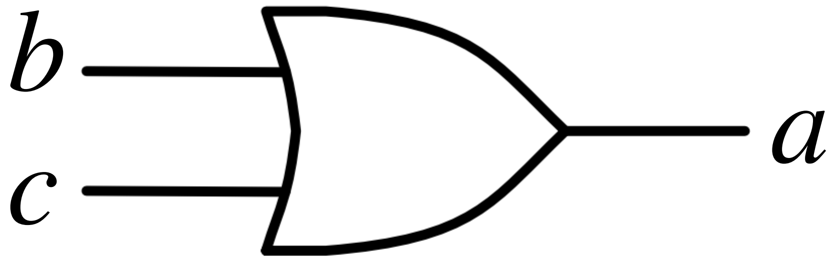
- After this operation, all gates have at most 2 wires feeding into them.

Reduction from CircuitSAT to 3SAT

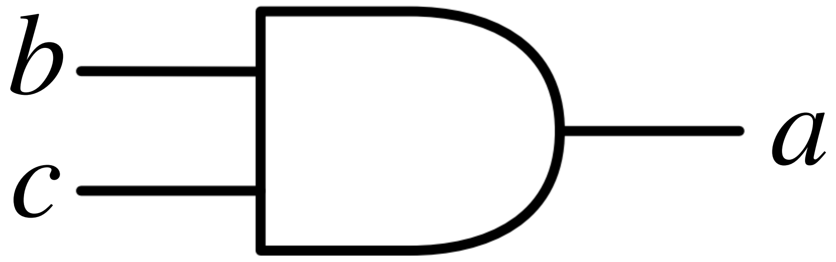
Step 2: Transform gates to formulas to clauses



$a = \overline{b}$ → $(a \vee b) \wedge (\overline{a} \vee \overline{b})$



$a = b \vee c$ → $(\overline{a} \vee b \vee c) \wedge (a \vee \overline{b}) \wedge (a \vee \overline{c})$



$a = b \wedge c$ → $(a \vee \overline{b} \vee \overline{c}) \wedge (\overline{a} \vee b) \wedge (\overline{a} \vee c)$

- Additionally, add the clause (z), indicating that the output wire of C is set to 1.

Reduction from CircuitSAT to 3SAT

Step 3: Fill up clauses of smaller width to obtain a 3CNF

$(a \vee b) \rightarrow (a \vee b \vee z) \wedge (a \vee b \vee \bar{z})$

$(a) \rightarrow (a \vee x \vee y) \wedge (a \vee \bar{x} \vee y) \wedge (a \vee x \vee \bar{y}) \wedge (a \vee \bar{x} \vee \bar{y})$

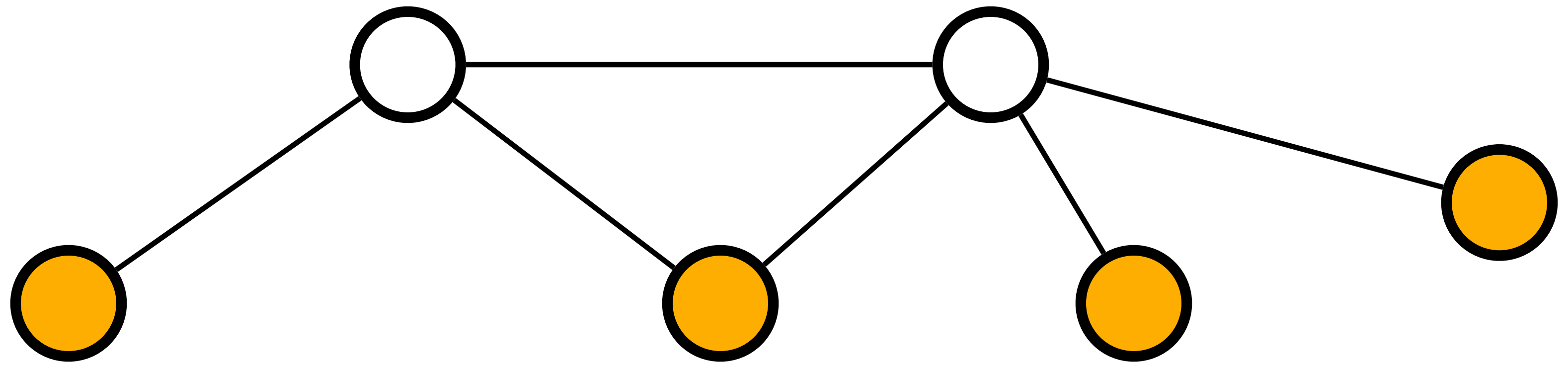
- After this operation, all clauses have exactly 3 literals.
- The entire reduction takes only polynomial time.
- **Result.** 3SAT is NP-hard.

Maximum Independent Set is NP-hard

Erickson, Section 12.7

Maximum Independent Set Problem

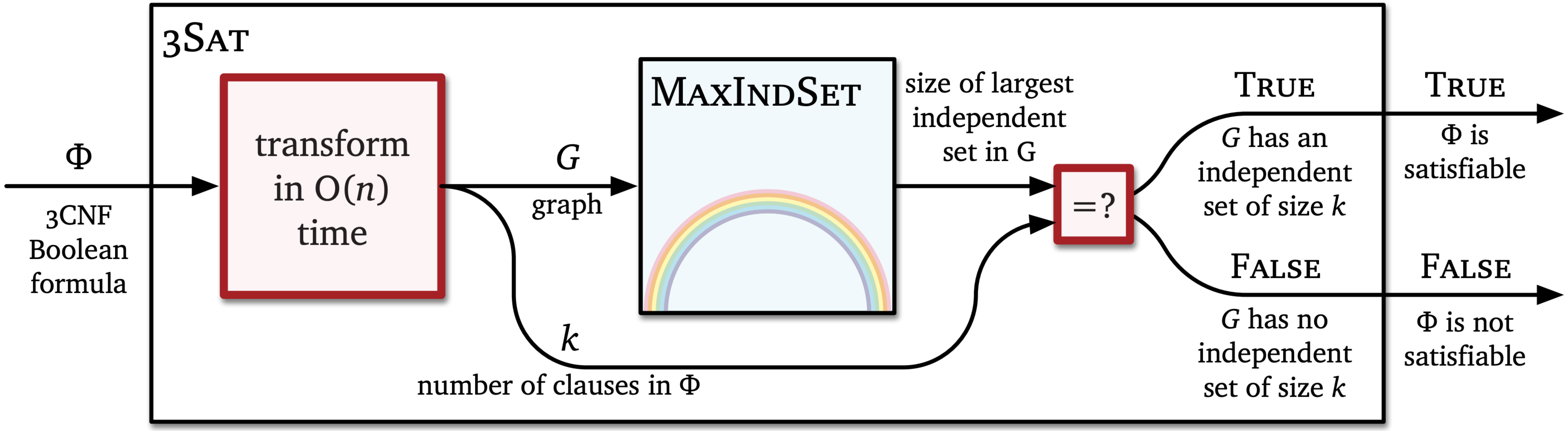
- **Independent Set.** Set S ⊆ V(G) such that no two vertices in S are adjacent



- **Maximum Independent Set Problem.**
  - **Input.** Graph G
  - **Output.** Size |S| of a maximum independent set S.

Reduction from 3SAT to MaxIndSet

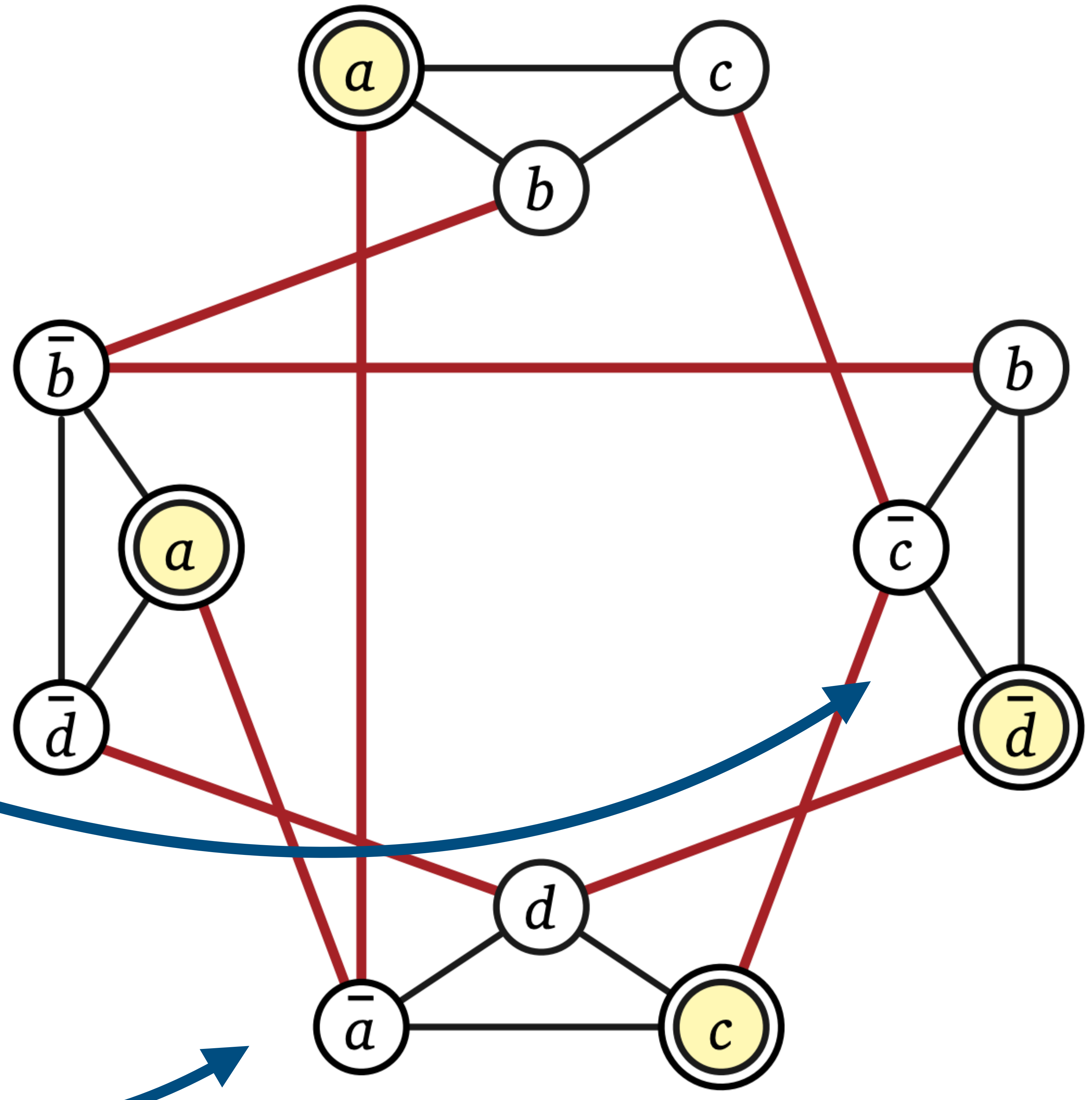
Overview



Reduction from 3SAT to MaxIndSet

Example

$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$



Given Φ, construct G as follows:

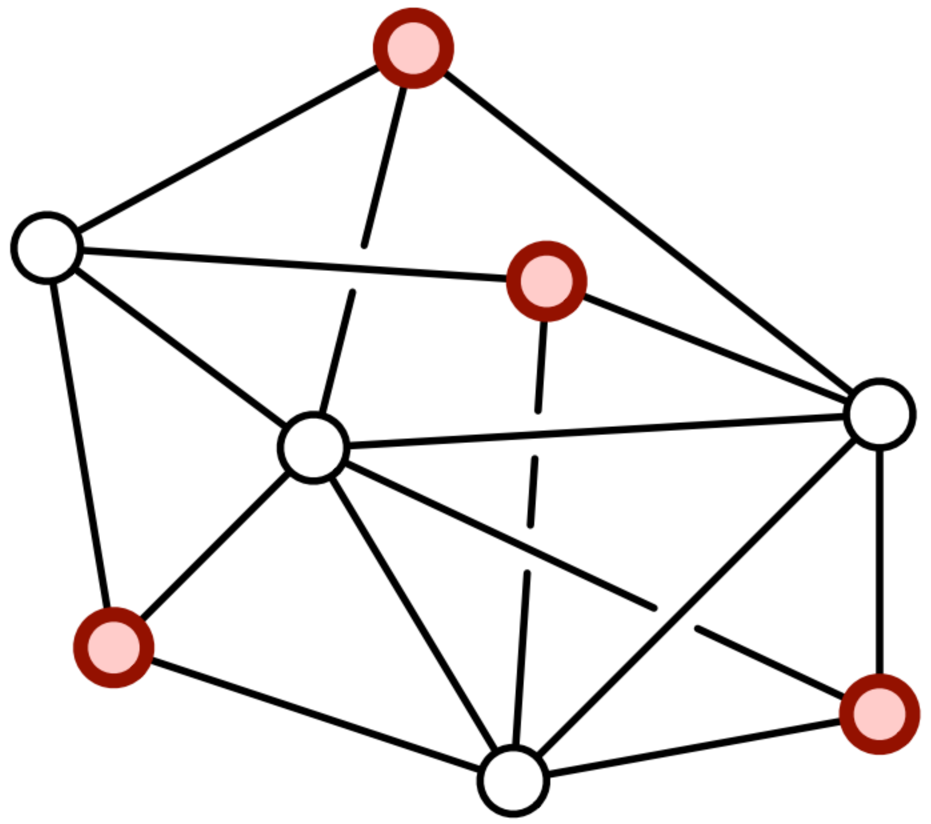
- create a triangle for each clause
- make inconsistent literals adjacent

Exercise. Why is this reduction correct?
Read the proof. [Erickson, Section 12.7]

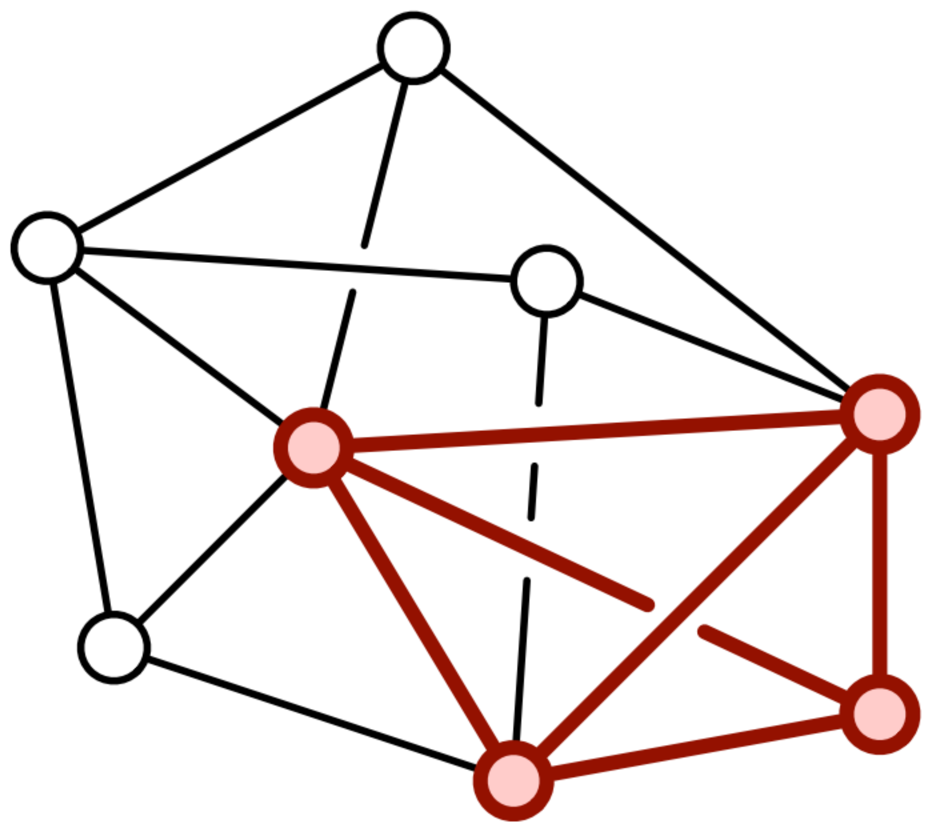
Maximum Clique is NP-hard
Minimum Vertex-Cover is NP-hard

Erickson, Section 12.9

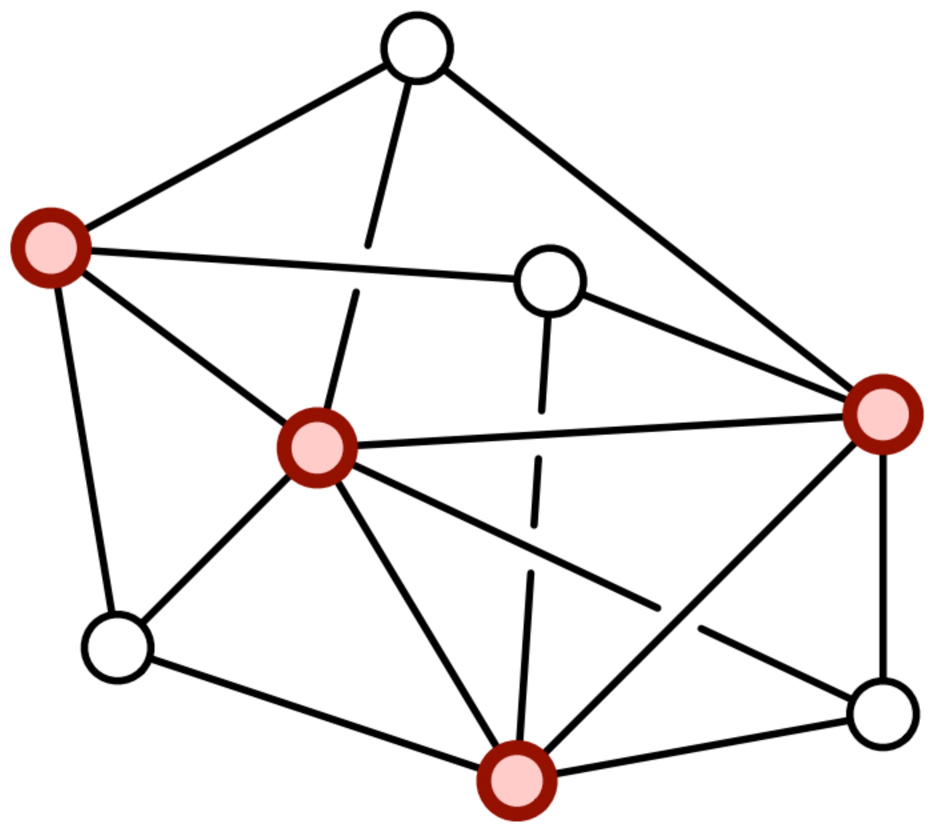
Three related problems



Maximum
Independent Set

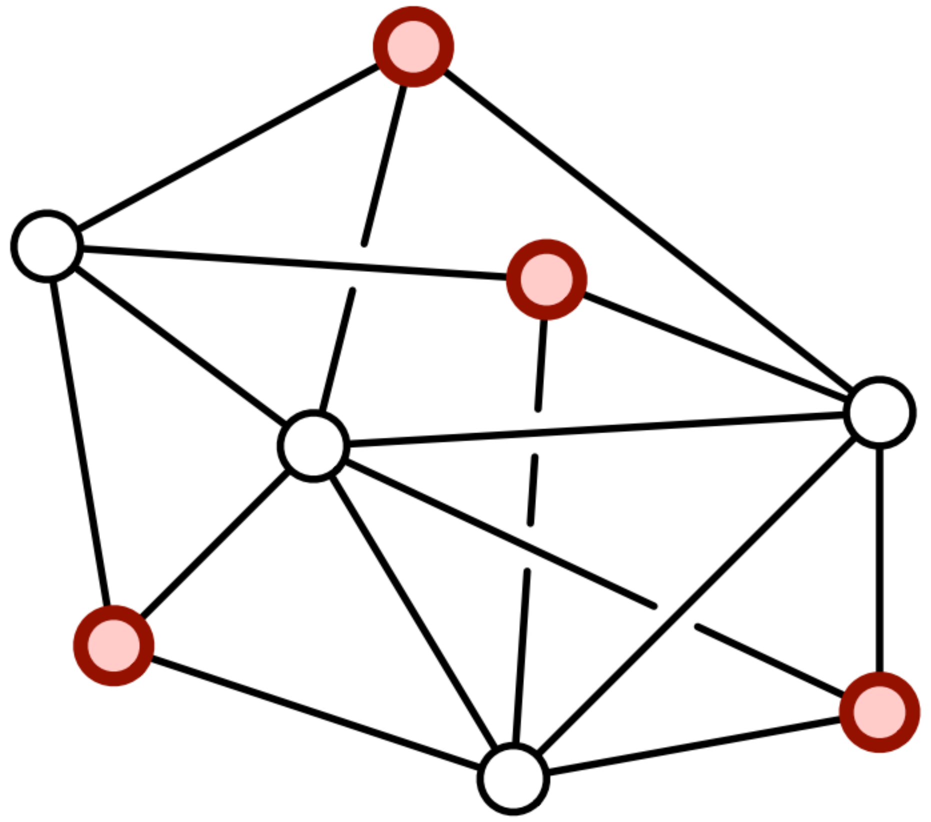


Maximum Clique

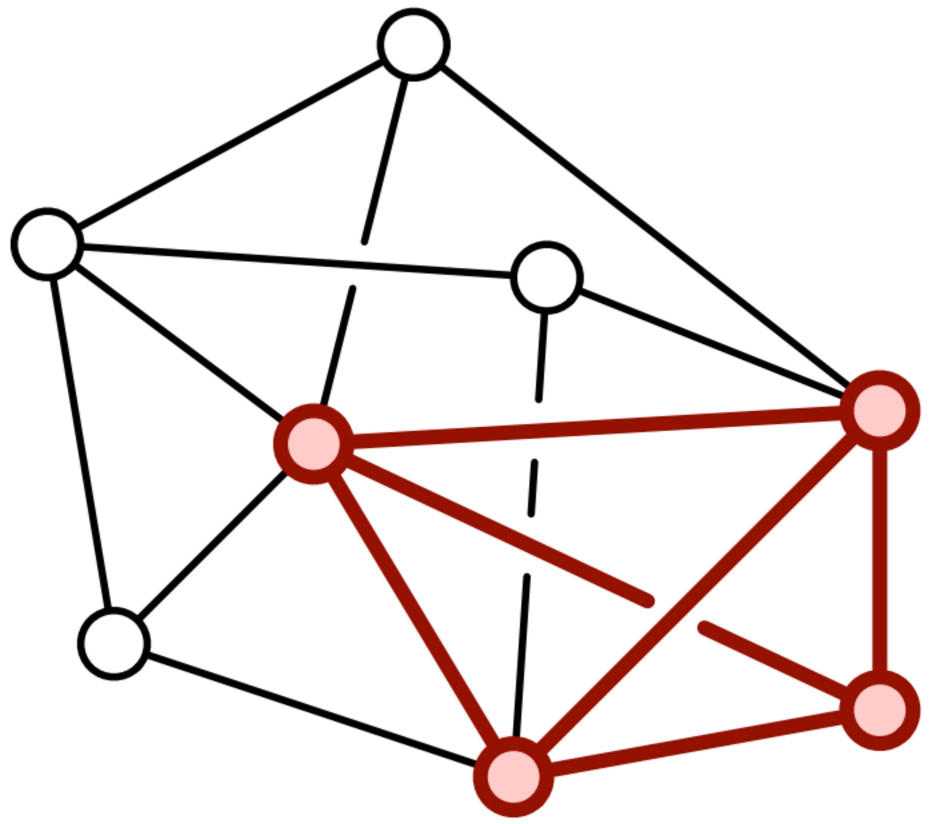


Minimum
Vertex-Cover

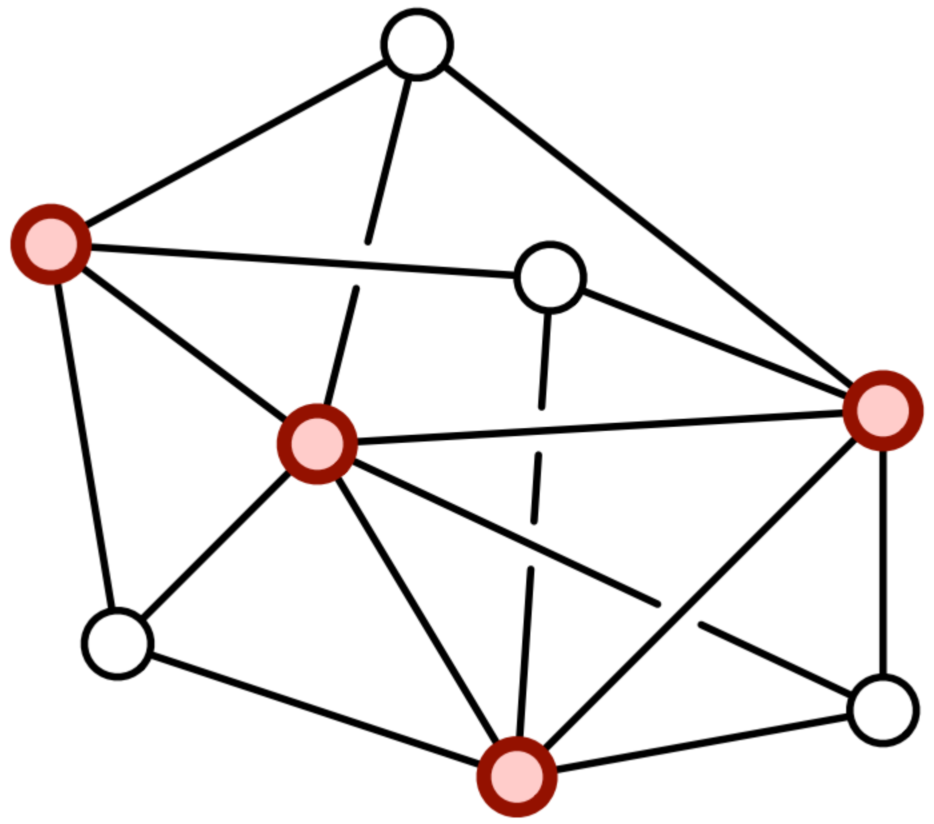
Three related problems



Maximum
Independent Set



Maximum Clique



Minimum
Vertex-Cover

Exercise. How exactly are
these two concepts related?

Independent Set vs Clique

graph G

complement graph $\overline{G}$

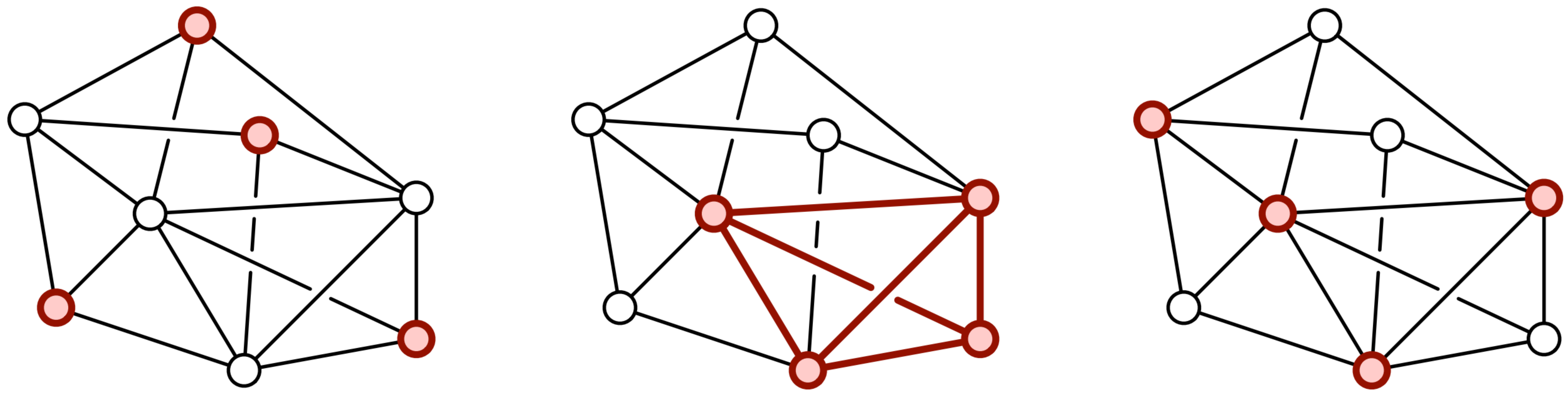
S is an independent sets of G

$\iff$ S contains no edges from G

$\iff$ S contains no non-edges from $\overline{G}$

$\iff$ S is a clique of $\overline{G}$

Three related problems



Maximum Independent Set

Maximum Clique

Minimum Vertex-Cover

Exercise. How exactly are these two concepts related?

Independent Set vs Vertex-Cover

S is an **independent sets** of G

$\Longleftrightarrow$ S contains no edges from G

$\Longleftrightarrow$ all edges of G intersect with V – S

$\Longleftrightarrow$ V – S is a **vertex-cover** of G

Polynomial-time Reductions