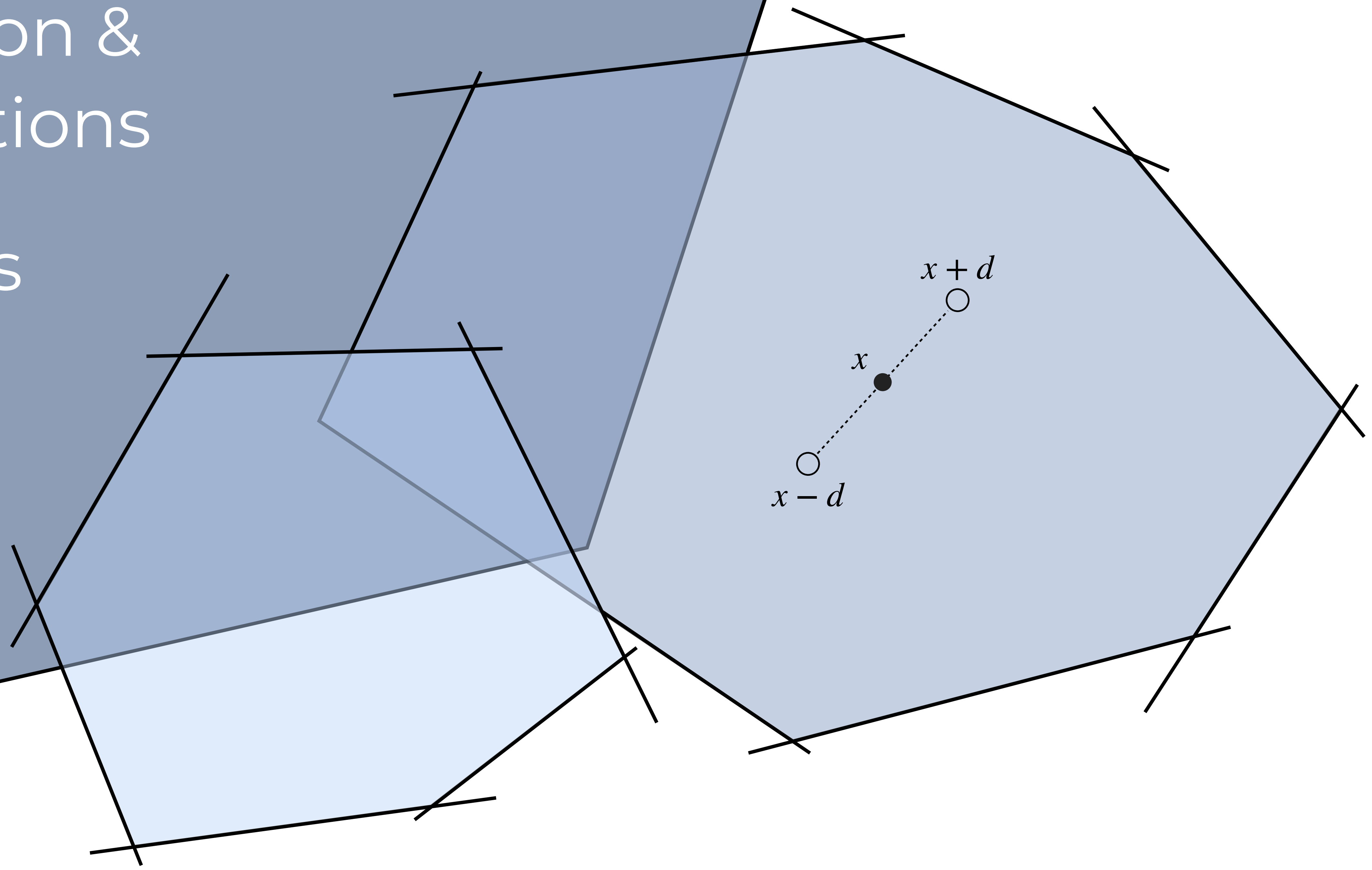
Leo Krull

Linear Programming I

LP Definition

Feasible Region & Optimal Solutions

LP Algorithms



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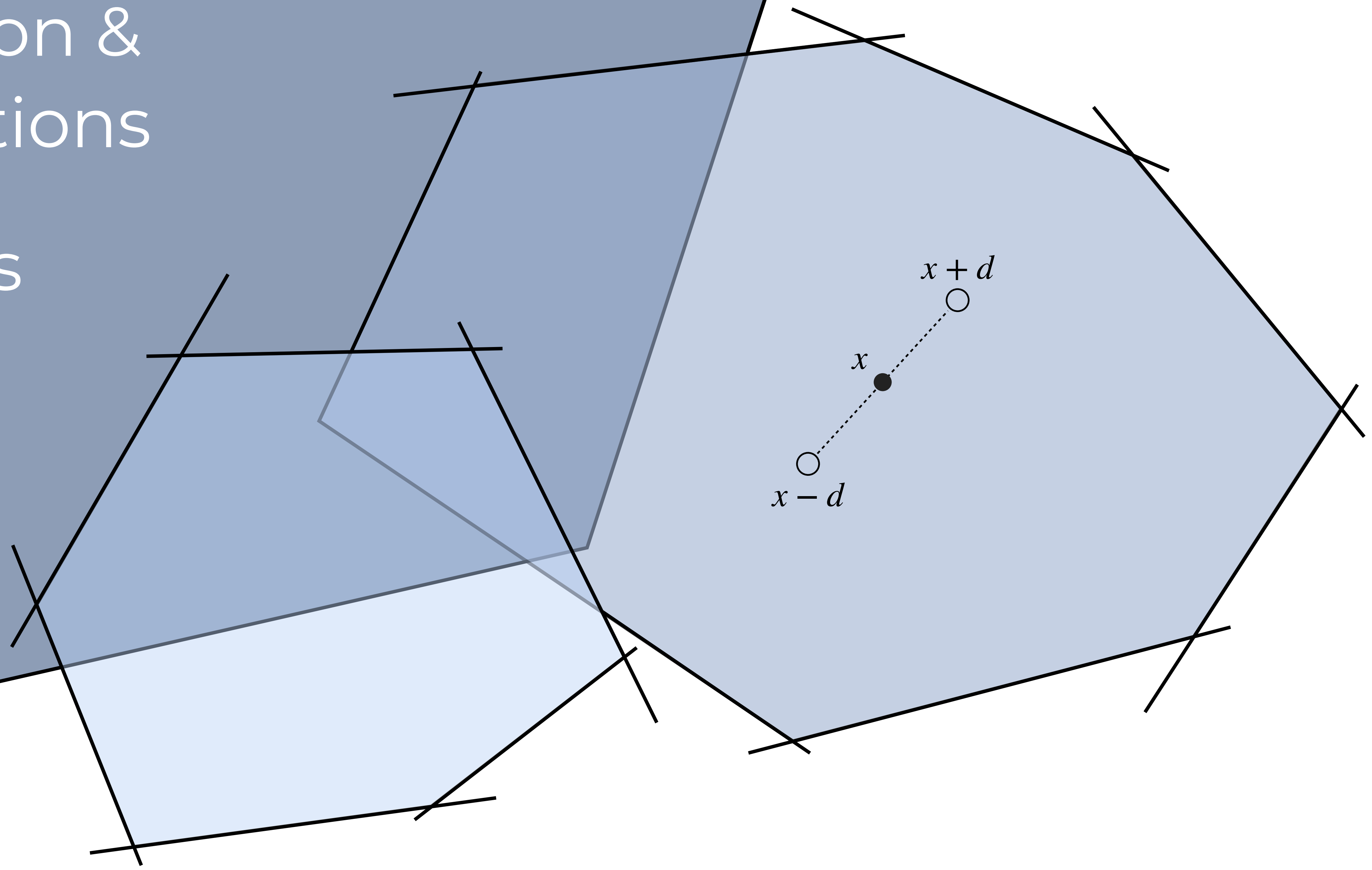
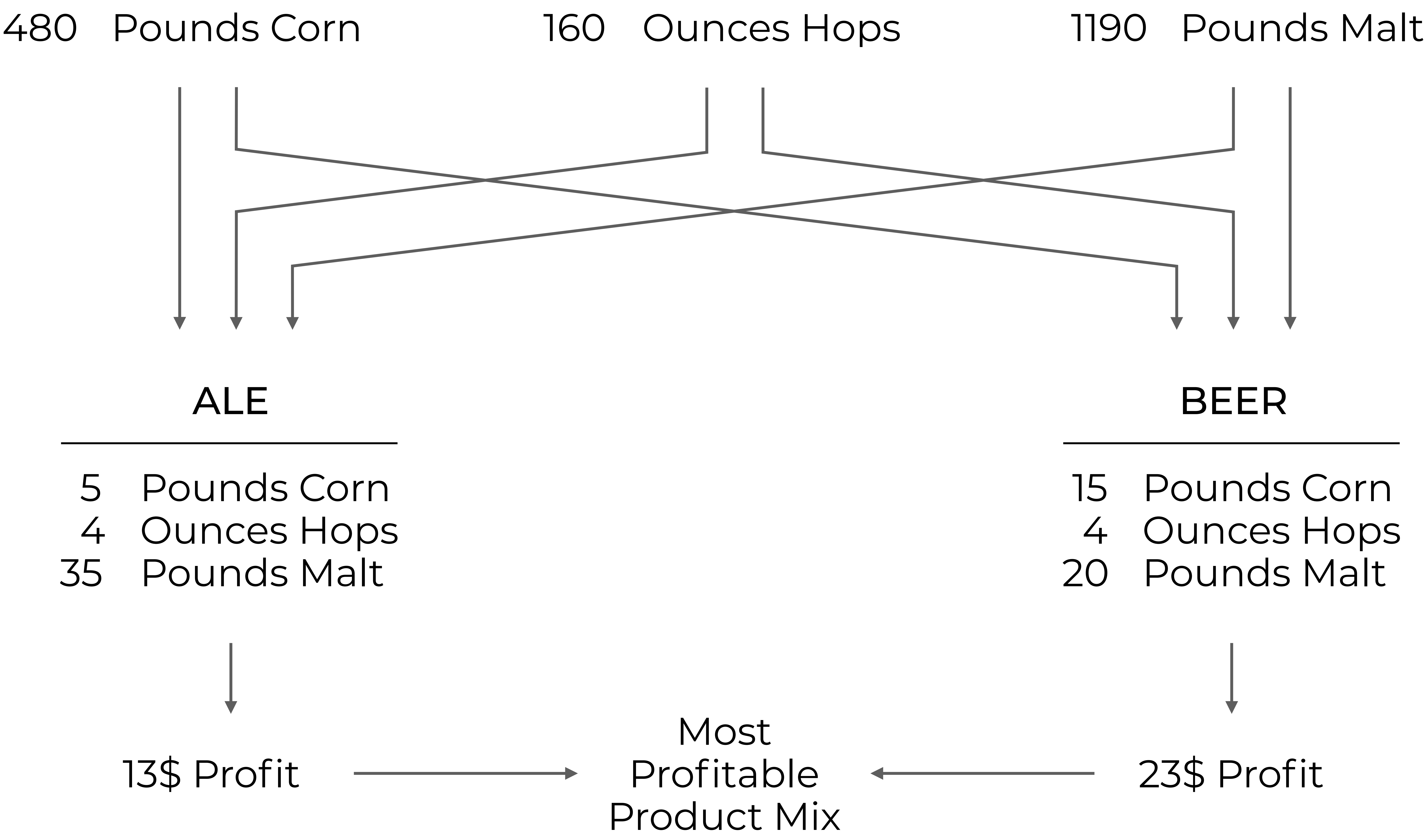


1 Brewery Example



480 Pounds Corn

160 Ounces Hops

1190 Pounds Malt

ALE

5 Pounds Corn
4 Ounces Hops
35 Pounds Malt

BEER

15 Pounds Corn
4 Ounces Hops
20 Pounds Malt

13$ Profit

Most Profitable Product Mix

23$ Profit

1 Brewery Example

	Ale	Beer	Quantity
Profit	13	23	
Corn	5	15	480
Hops	4	4	160
Malt	35	20	1190

1 Brewery Example: Decision Variables

	Decision Variables		Quantity
Profit	$13\ x_1$	$23\ x_2$	
Corn	$5\ x_1$	$15\ x_2$	480
Hops	$4\ x_1$	$4\ x_2$	160
Malt	$35\ x_1$	$20\ x_2$	1190
	$x_1,$	x_2	

1 Brewery Example: Constraints

$$
\begin{array}{lrlcrlcr}
\text{Profit} & 13 & x_1 & & 23 & x_2 & & \\
 & 5 & x_1 & + & 15 & x_2 & \leq & 480 \\
\text{Constraints} & 4 & x_1 & + & 4 & x_2 & \leq & 160 \\
 & 35 & x_1 & + & 20 & x_2 & \leq & 1190 \\
 & & x_1 & , & & x_2 & \geq & 0
\end{array}
$$

1 Brewery Example: Objective Function

Objective Function

$$\begin{array}{rrcrrcr}
\max & 13\,x_1 & + & 23\,x_2 & & & \\
 & 5\,x_1 & + & 15\,x_2 & \leq & & 480 \\
 & 4\,x_1 & + & 4\,x_2 & \leq & & 160 \\
 & 35\,x_1 & + & 20\,x_2 & \leq & & 1190 \\
 & x_1 & , & x_2 & \geq & & 0
\end{array}$$

1 Brewery Example: LP Formulation

$$
\begin{array}{rrcrcr}
\max\ 13\ x_1 & + & 23\ x_2 & & \\
5\ x_1 & + & 15\ x_2 & \leq & 480 \\
4\ x_1 & + & 4\ x_2 & \leq & 160 \\
35\ x_1 & + & 20\ x_2 & \leq & 1190 \\
x_1 & , & x_2 & \geq & 0
\end{array}
$$

1 Brewery Example: Canonical Form

$$\max \quad (13 \quad 23)^T \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 15 \\ 4 & 4 \\ 35 & 20 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \leq \quad \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$

$$x_1, x_2 \quad \geq \quad 0$$

1 Brewery Example: LP Terminology

$$
\begin{array}{rcccl}
 & \max & \overbrace{(13\quad 23)^T}^{\substack{\text{objective vector}\\ c\in\mathbb{R}^2}}\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix} & & \\
\substack{\text{constraint matrix}\\ A\in\mathbb{R}^{3\times 2}}\left\{\begin{pmatrix}5&15\\4&4\\35&20\end{pmatrix}\right. & \cdot\begin{pmatrix}x_1\\x_2\end{pmatrix} & \le & \left.\begin{pmatrix}480\\160\\1190\end{pmatrix}\right\}\substack{\text{right-hand side}\\ b\in\mathbb{R}^3} & \\
 & \underbrace{x_1, x_2}_{\substack{\text{solution vector}\\ x\in\mathbb{R}^2_{\ge 0}}} & \ge & 0 &
\end{array}
$$

1 Standard Form of an LP

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, then a linear program (LP) in standard form is given by

$$\begin{aligned} &\max c^T x \\ &A x = b \\ &x \geq 0 \end{aligned}$$

$$\begin{aligned} &\max \sum_{j=1}^{n} c_j x_j & \\ &\sum_{j=1}^{n} a_{ij} x_j = b_i & 1 \leq i \leq m \\ &x_j \geq 0 & 1 \leq j \leq n \end{aligned}$$

A linear program asks for a vector $x \in \mathbb{R}^n_{\geq 0}$ that maximizes or minimizes a given linear function among all vectors x that satisfy a given set of linear inequalities.

1 Brewery Example: Converting to Standard Form

Canonical Form

$$
\begin{aligned}
\max \quad & 13x_1 + 23x_2 \\
& 5x_1 + 15x_2 \le 480 \\
& 4x_1 + 4x_2 \le 160 \\
& 35x_1 + 20x_2 \le 1190 \\
& x_1, \quad x_2 \ge 0
\end{aligned}
$$

Standard Form

$$
\begin{aligned}
\max \quad & 13x_1 + 23x_2 \\
& 5x_1 + 15x_2 + s_1 = 480 \\
& 4x_1 + 4x_2 + s_2 = 160 \\
& 35x_1 + 20x_2 + s_3 = 1190 \\
& x_1, \quad x_2, \quad s_1, \quad s_2, \quad s_3 \ge 0
\end{aligned}
$$

Obtain standard or slack form by adding one slack variable for each inequality.
Thus, the brewery LP in standard form is a 5-dimensional problem.

1 Converting to Standard Form

Different LP formulations can be converted into standard form:

Less than	⇒	Equality	$x + 2y - 3z \leq 17$	$\Rightarrow$	$x + 2y - 3z + s = 17,\ \ s \geq 0$
Greater than	⇒	Equality	$x + 2y - 3z \geq 17$	$\Rightarrow$	$x + 2y - 3z - s = 17,\ \ s \geq 0$
Min	⇒	Max	$\min\ \ x + 2y - 3z$	$\Rightarrow$	$\max\ \ -x - 2y + 3z$
Unrestricted	⇒	Nonnegative	x	$\Rightarrow$	$x = x^+ - x^-,\ \ x^+ \geq 0,\ \ x^- \geq 0$

1 Linear Programming

Linear Programming. Optimize a linear function subject to linear inequalities.

Generalizes

Shortest Path Problem
Max Flow
Assignment Problem
Matching
MST

Real-world Applications

Planning
Routing
Scheduling
Assignment

Ranked among most important scientific advances of the 20th century!

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LP Definition

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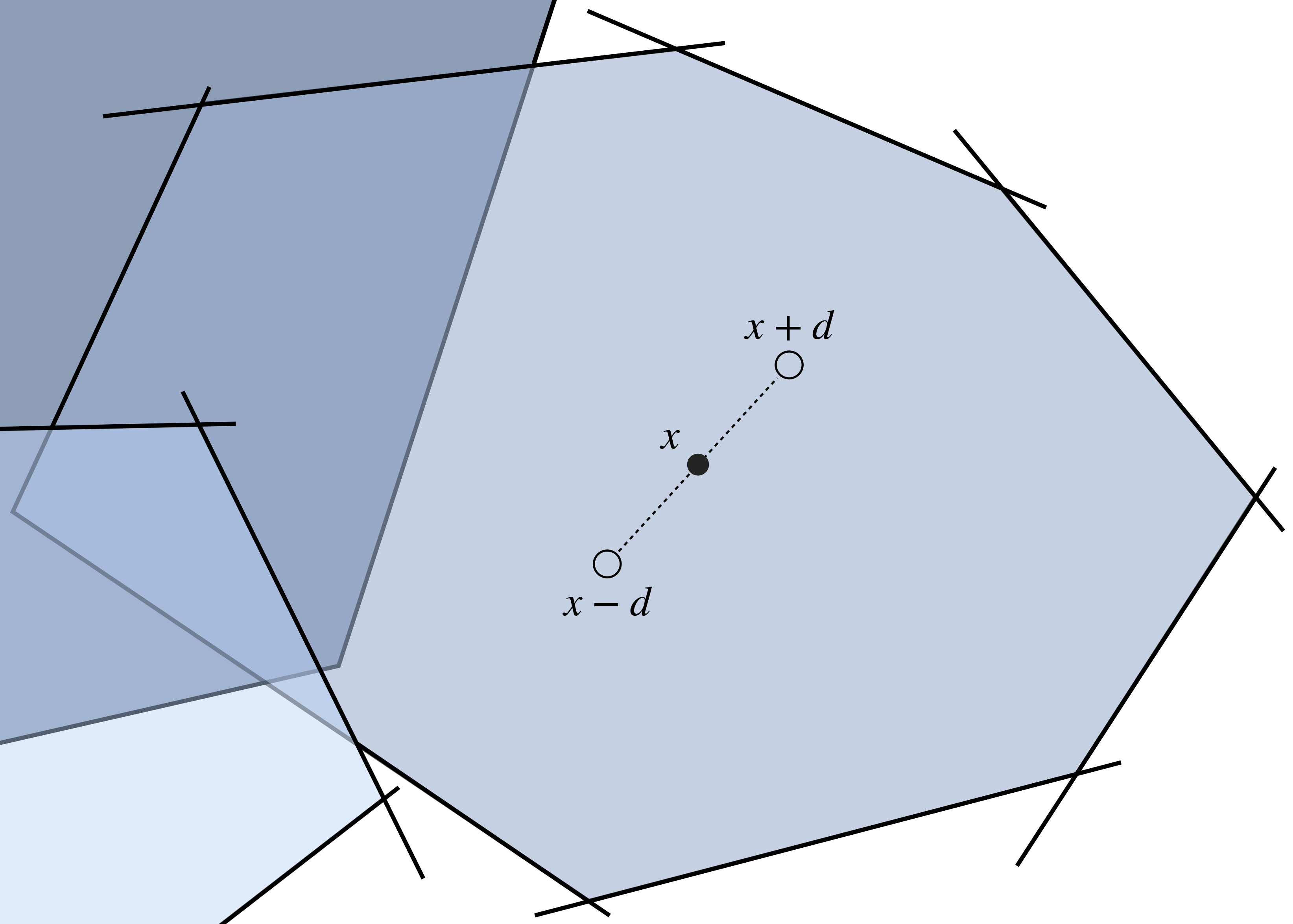
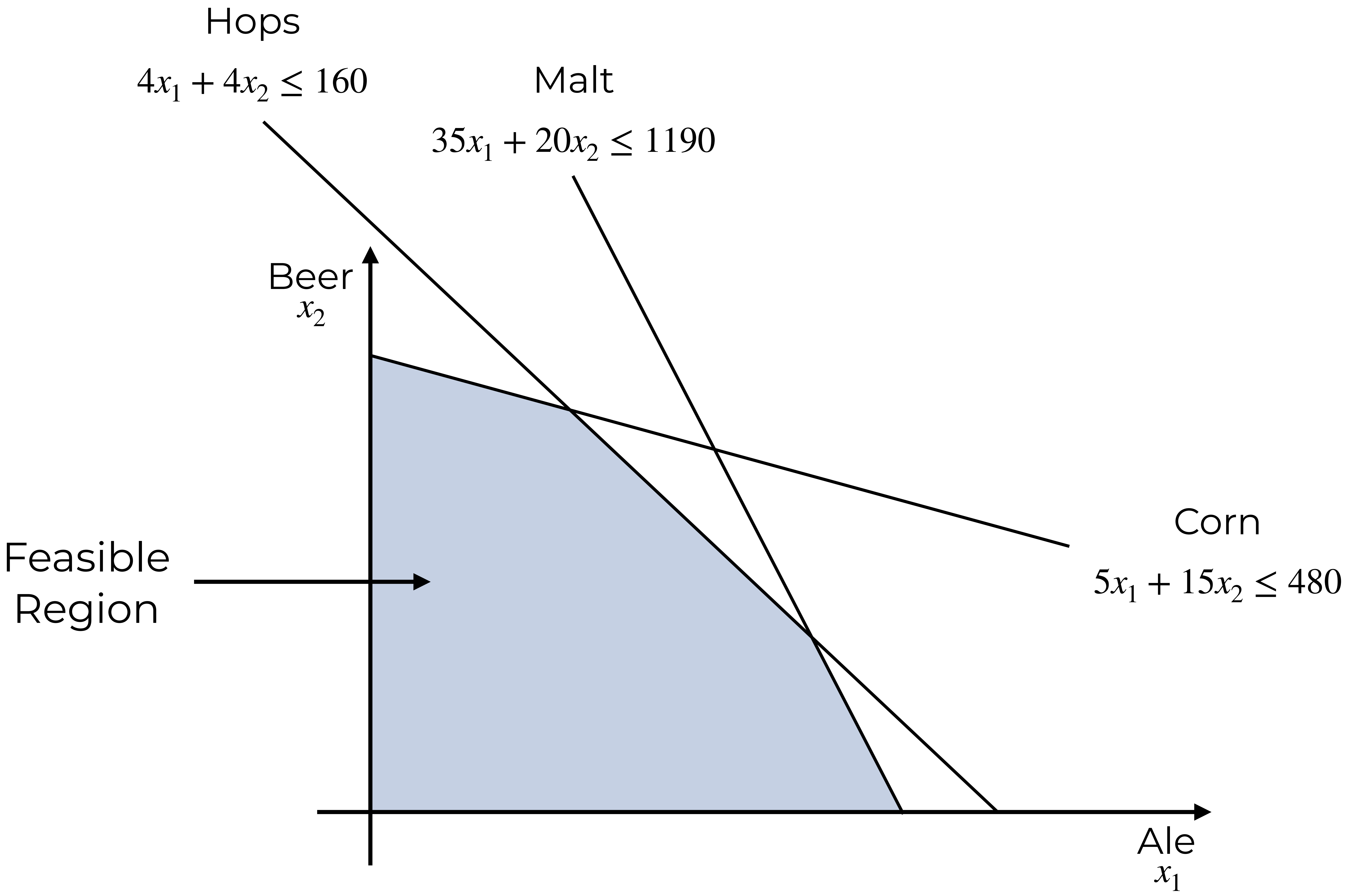
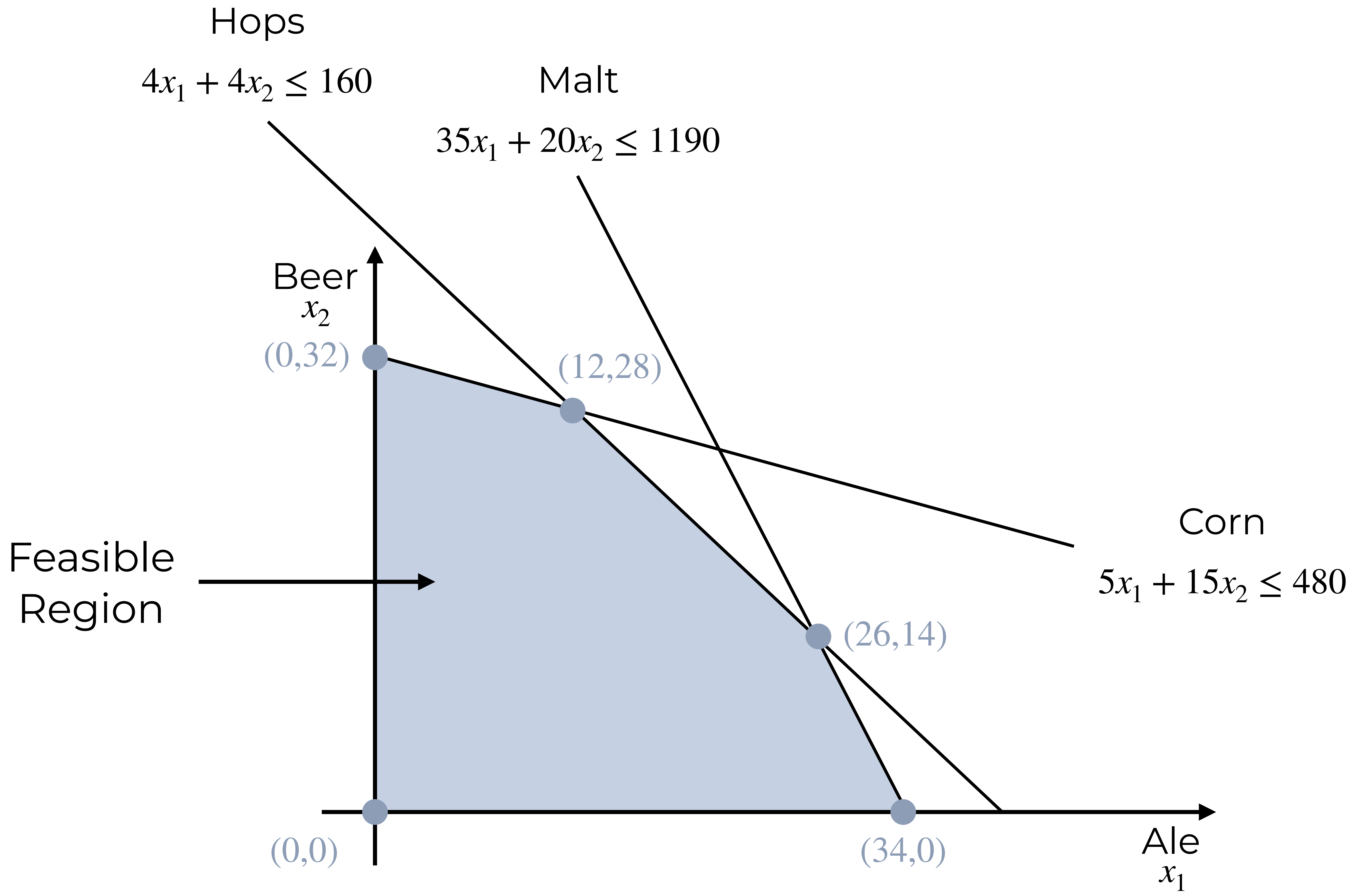
LP Algorithms



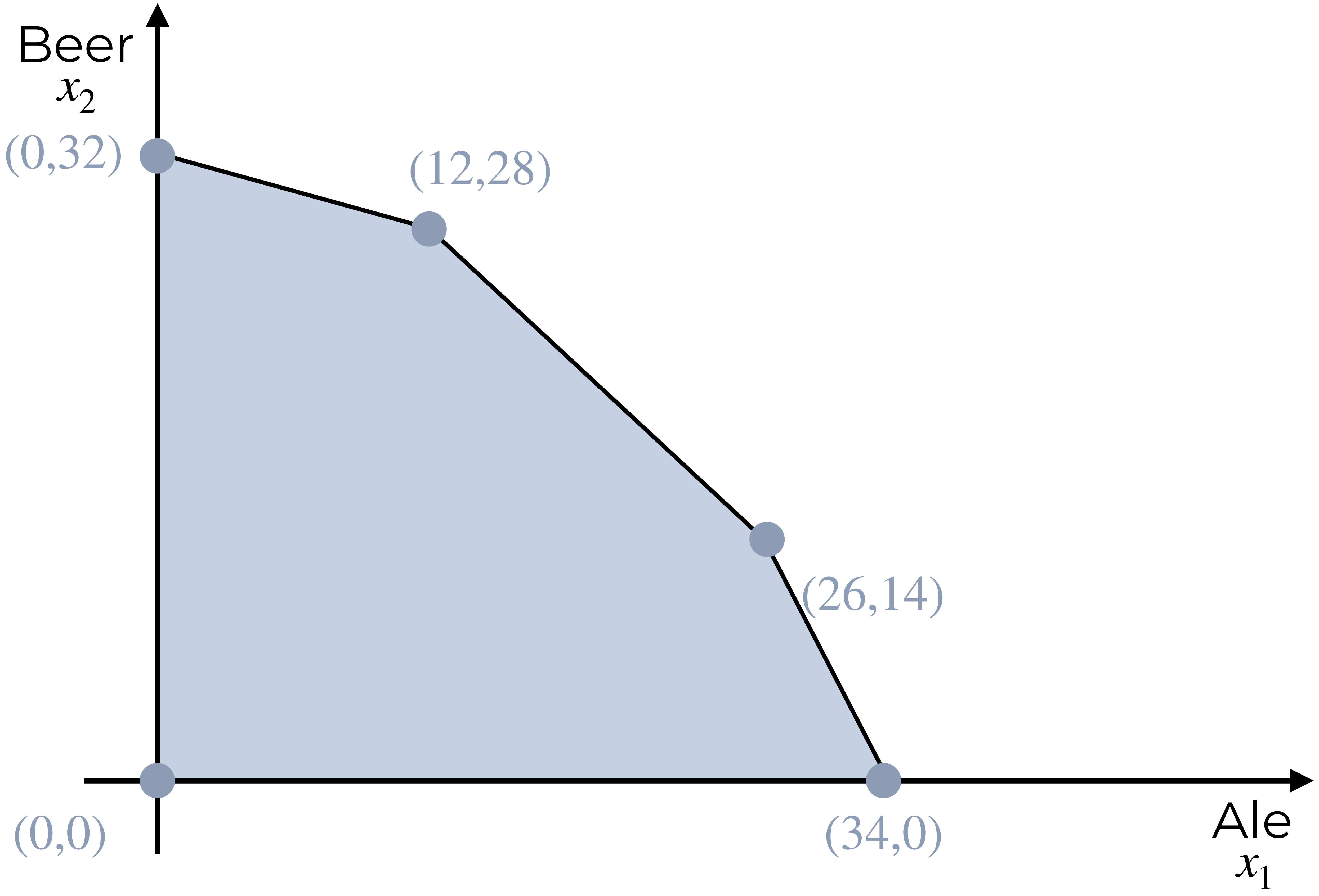
2 Brewery Example: Feasible Region



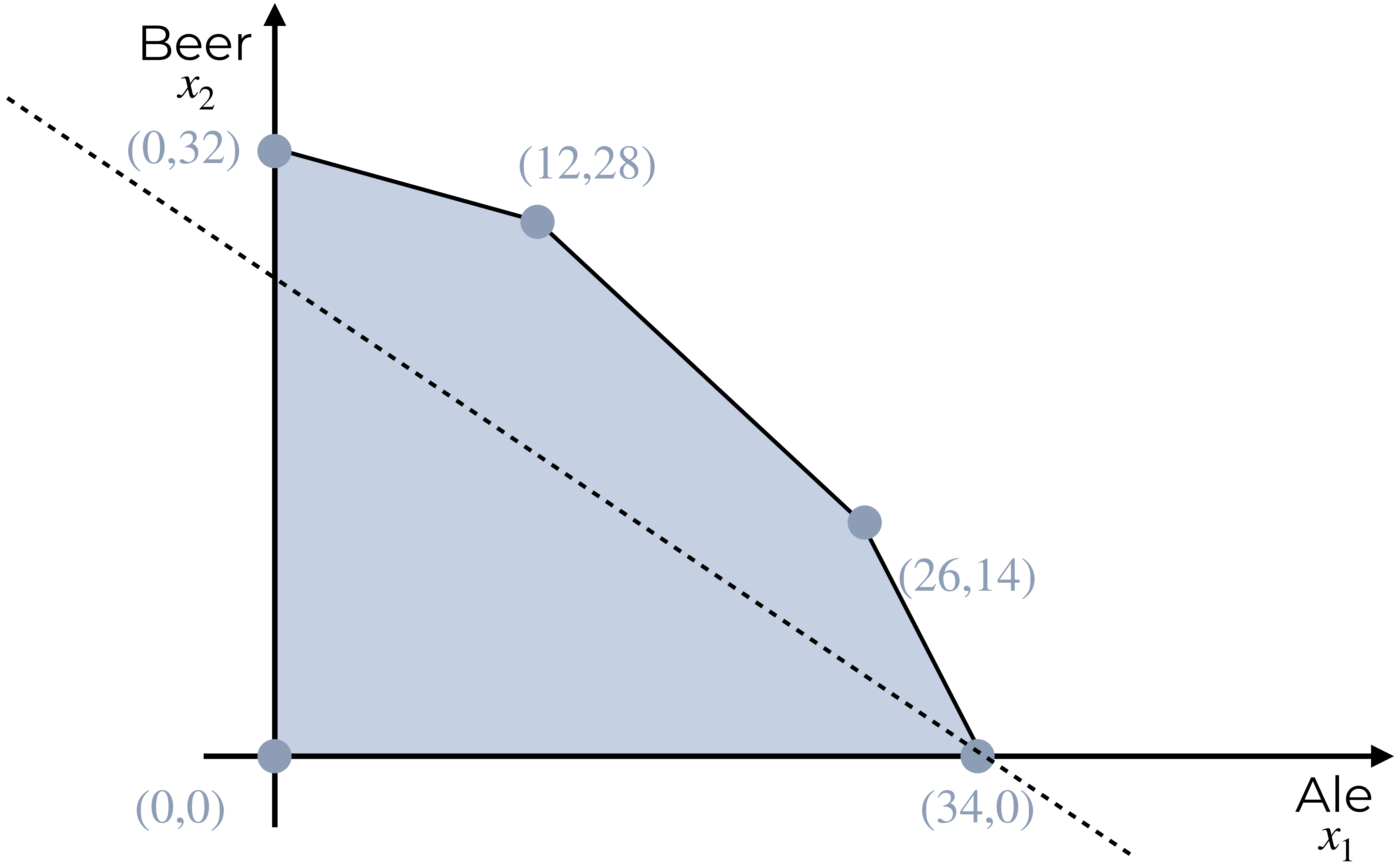
2 Brewery Example: Feasible Region



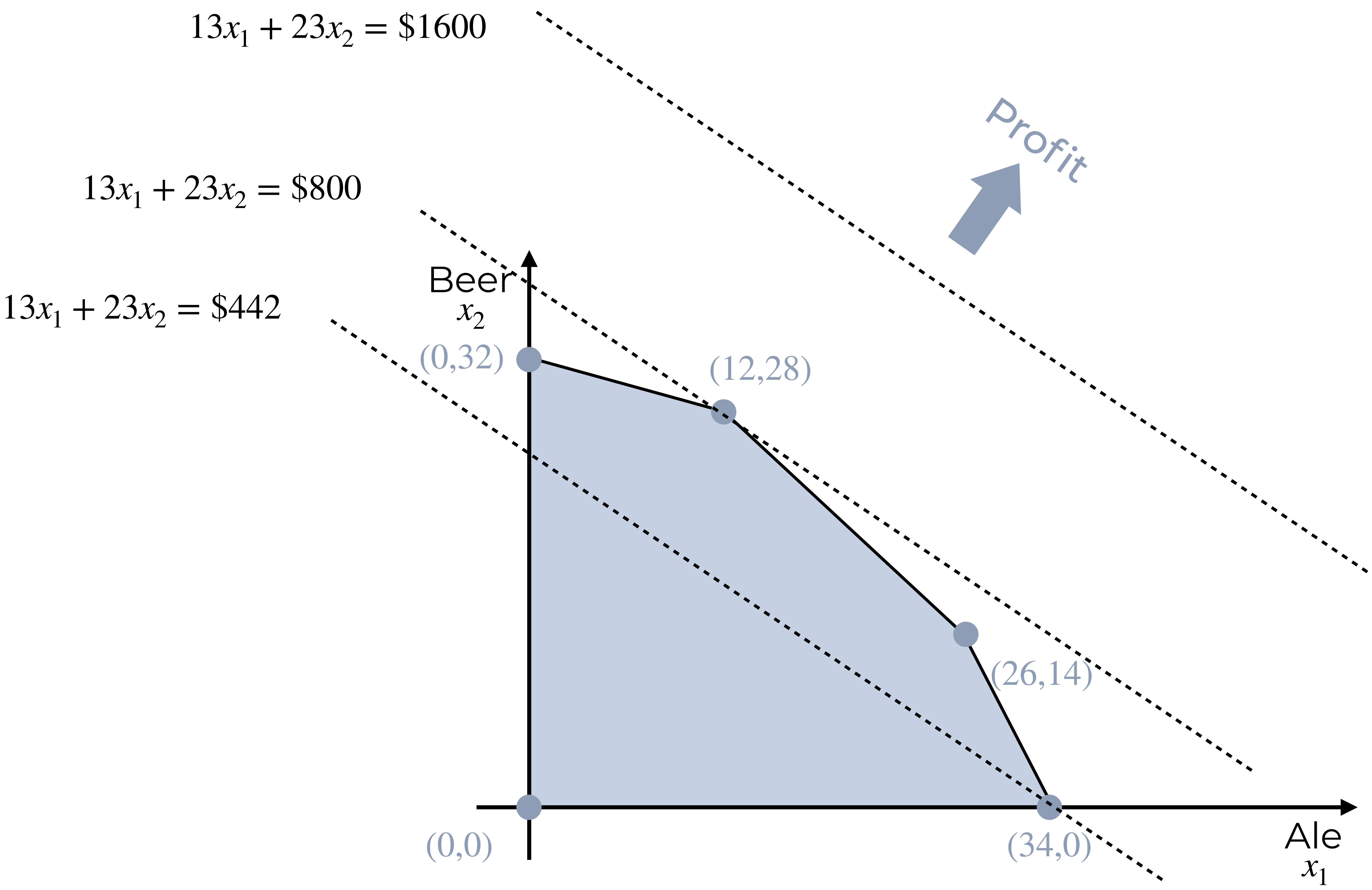
2 Brewery Example: Feasible Region



2 Brewery Example: Feasible Region

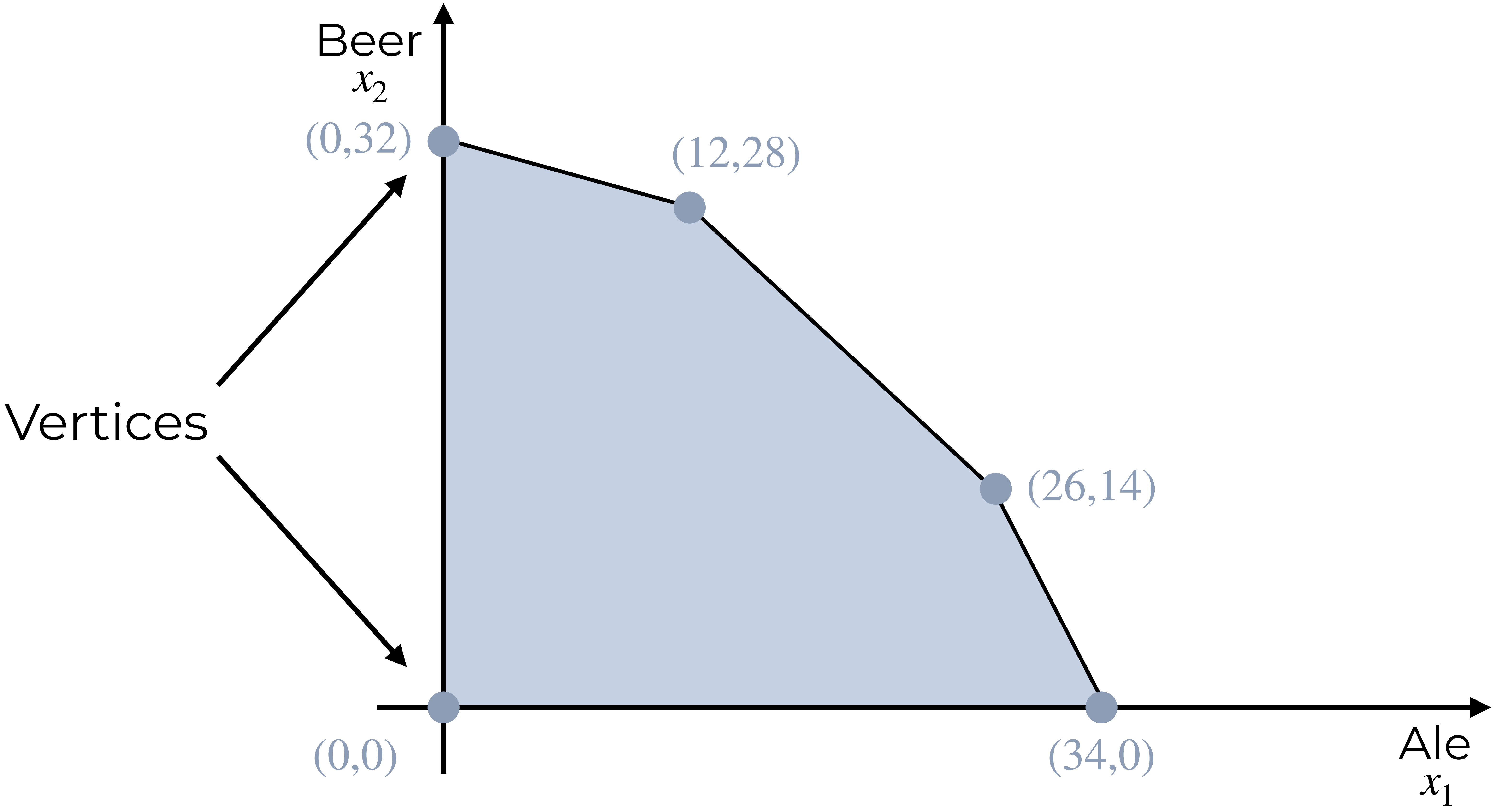


2 Brewery Example: Feasible Region



2 Brewery Example: Feasible Region

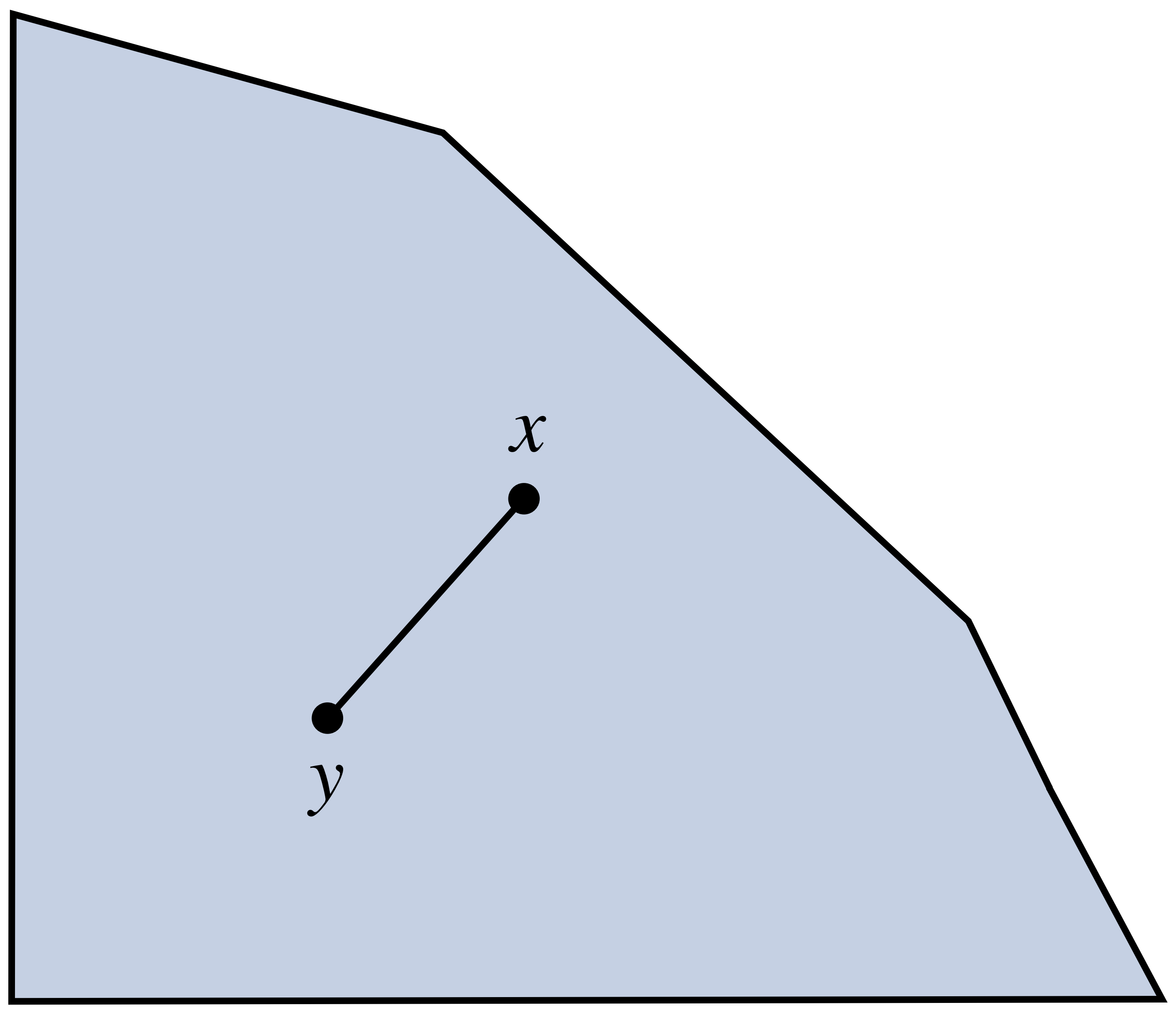
Observation. Regardless of the objective function coefficients, an optimal solution occurs at a vertex.



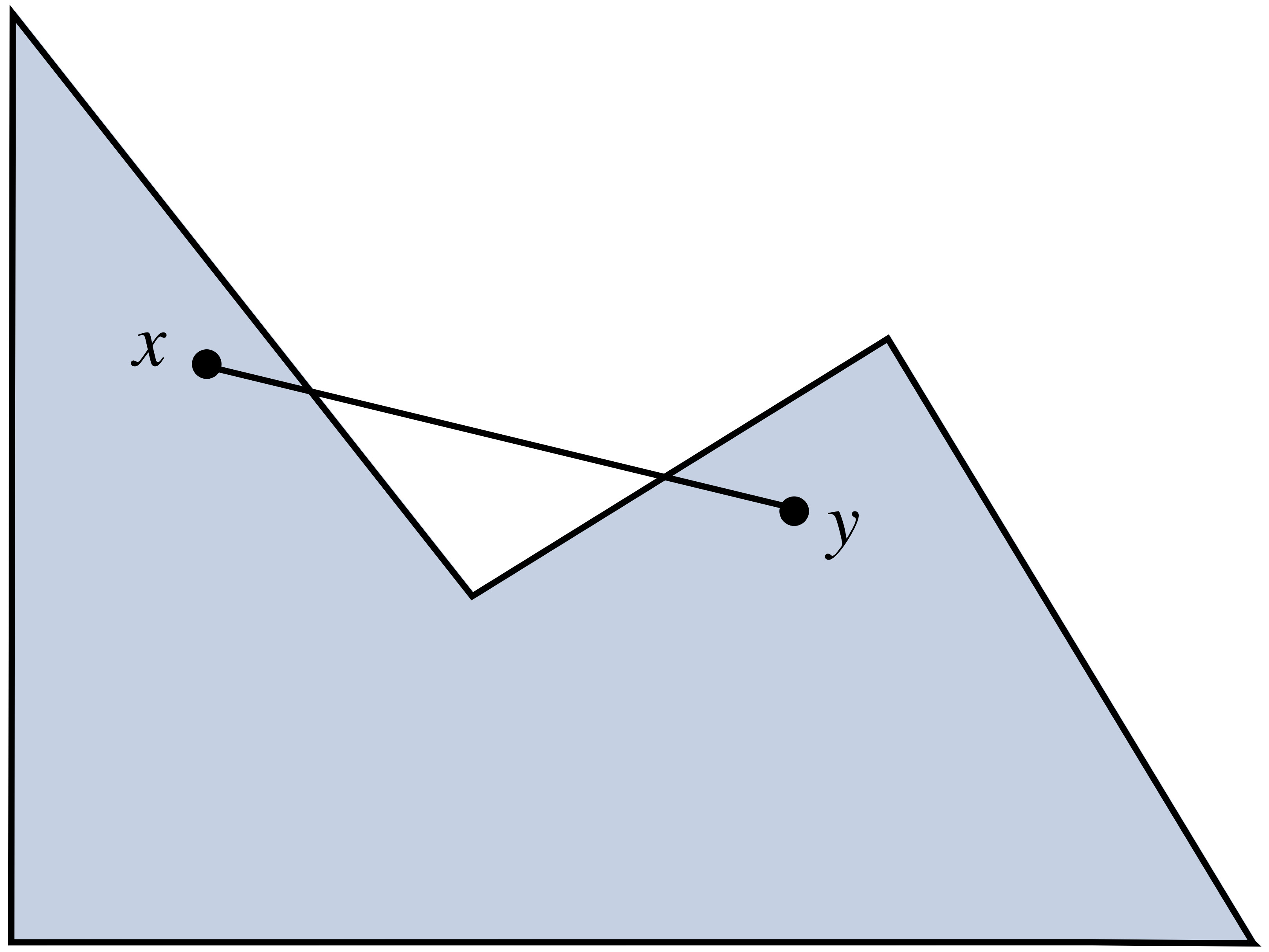
2 Geometry: Convexity

Convex Combination. Given the points $a^{(1)}, a^{(2)}, \dots, a^{(k)} \in \mathbb{R}^n$, a convex combination is $\sum_i \lambda_i a^{(i)}$ where $\lambda_i \geq 0$ for all i and $\sum_i \lambda_i = 1$.

Convex Set. If two points x and y are in the set, then so is $\lambda x + (1 - \lambda)y$ for $0 \leq \lambda \leq 1$.



Convex

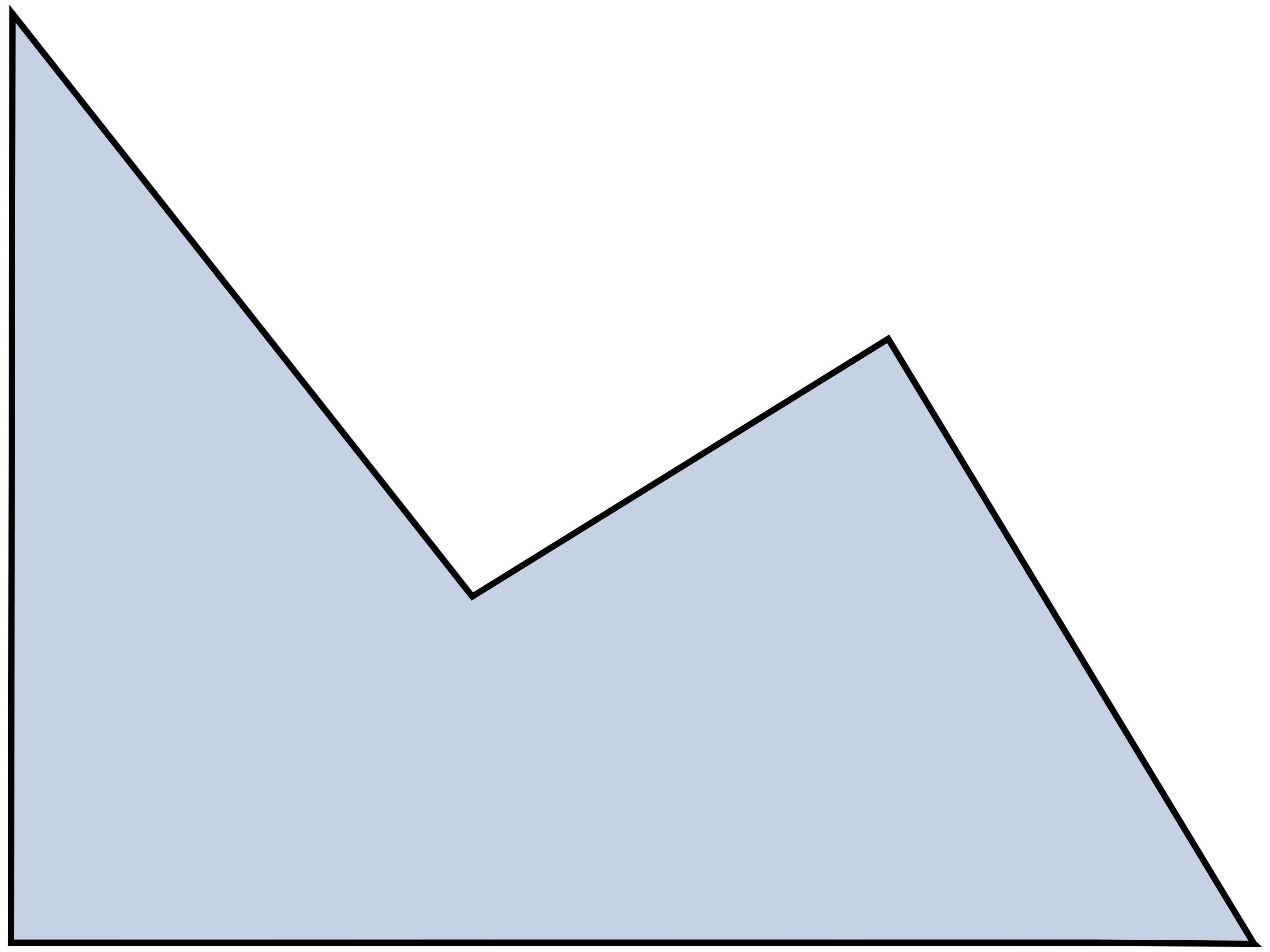


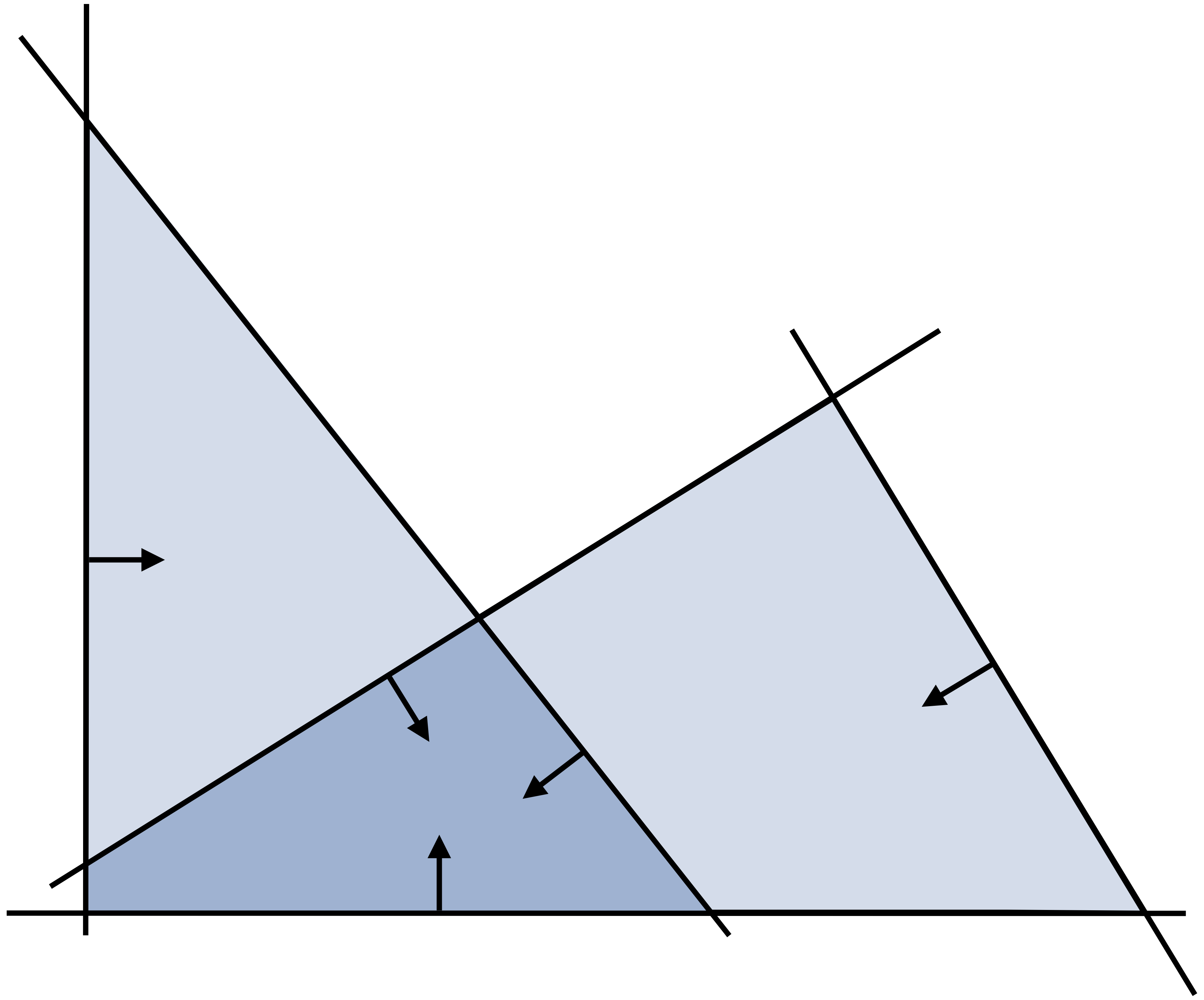
Not Convex

2 Geometry: Convexity

Convex Set. If two points x and y are in the set, then so is $\lambda x + (1 - \lambda)y$ for $0 \leq \lambda \leq 1$.

Observation. The feasible region of an LP is a convex set.

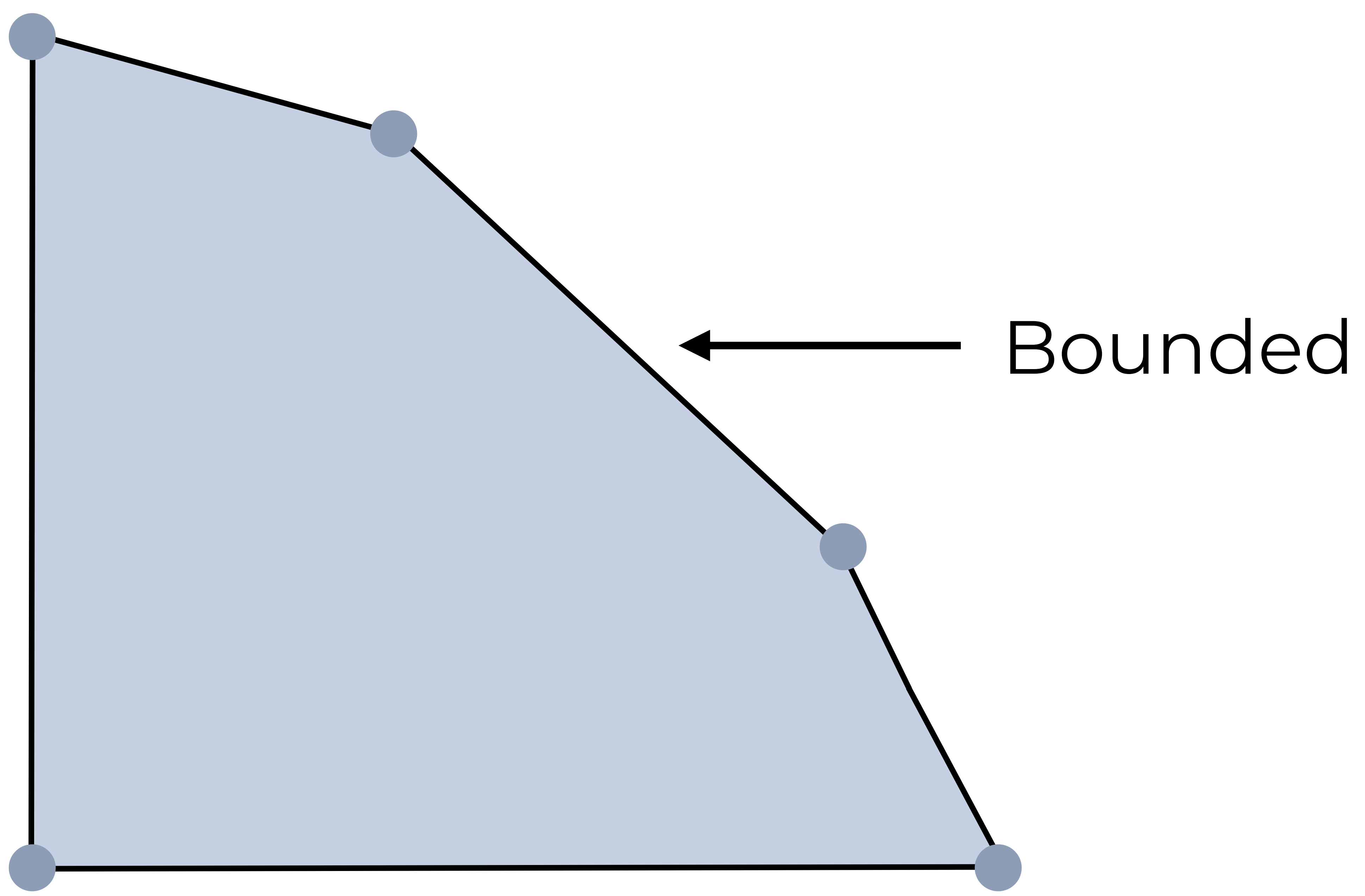




2 Geometry: Convexity

Convex Hull. The set of all convex combinations of elements of a set S is called the convex hull of S.

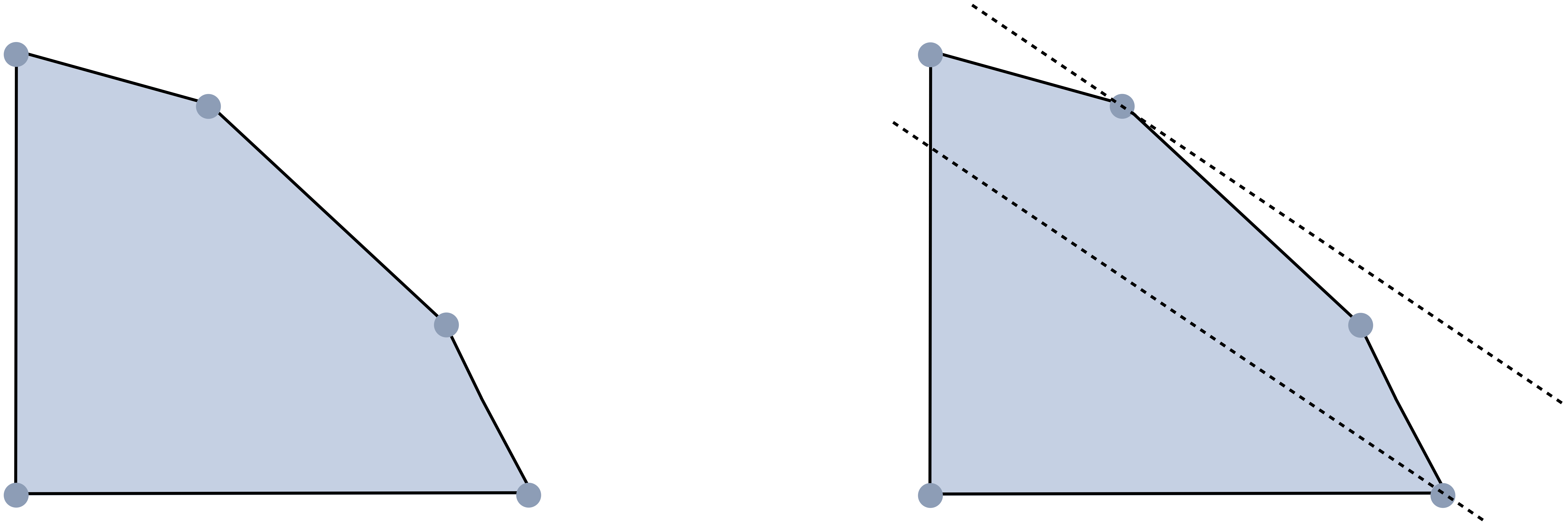
Polytope. A polytope is the convex hull of a finite set of points.



2 Geometry: Convexity

Vertex. A point x in a set S that cannot be written as a strict convex combination of two distinct points in S, i.e. $\exists d \neq 0 : x \pm d \in S$.

Theorem. If there exists an optimal solution to an LP, then there exists one that is a vertex.



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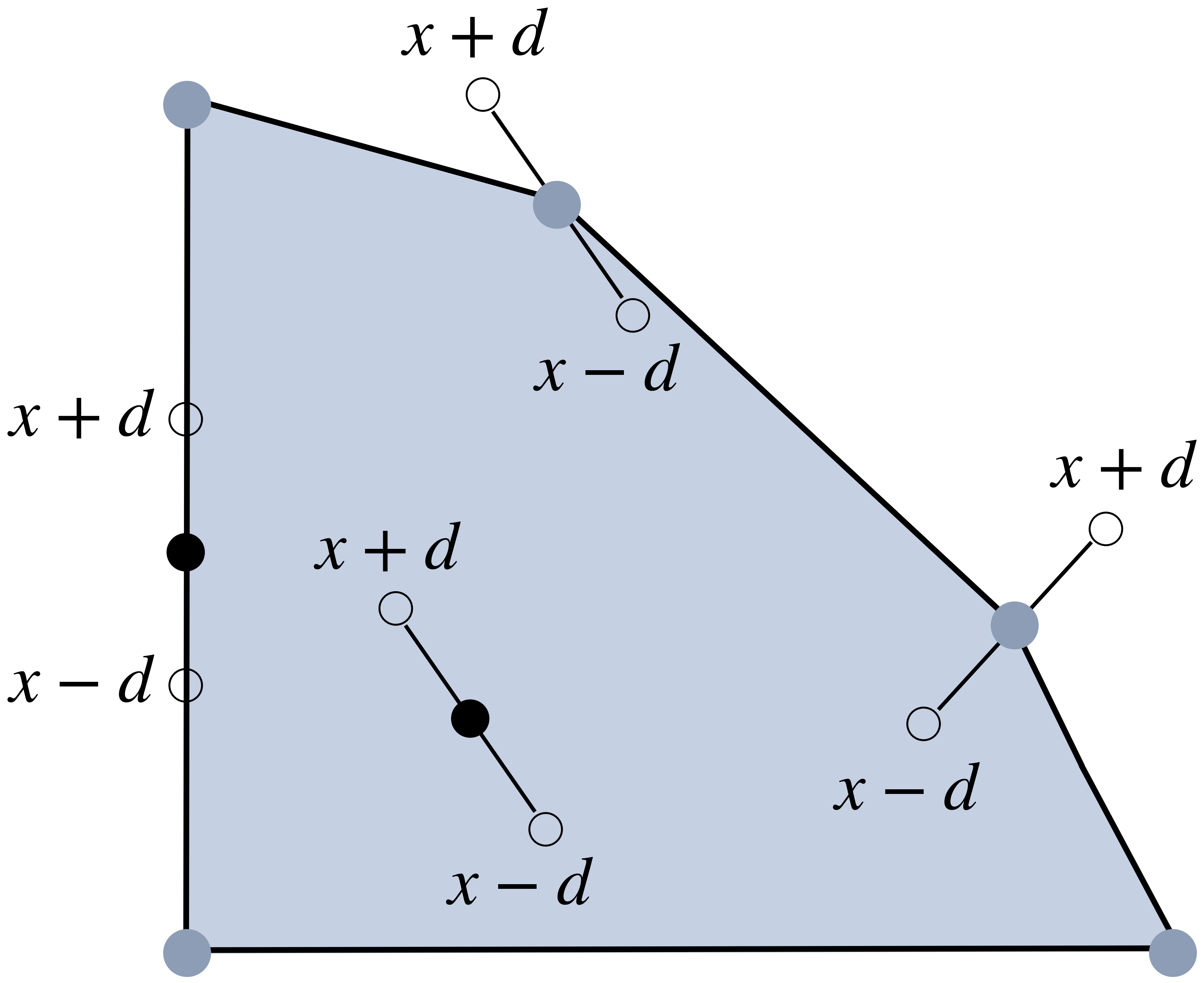
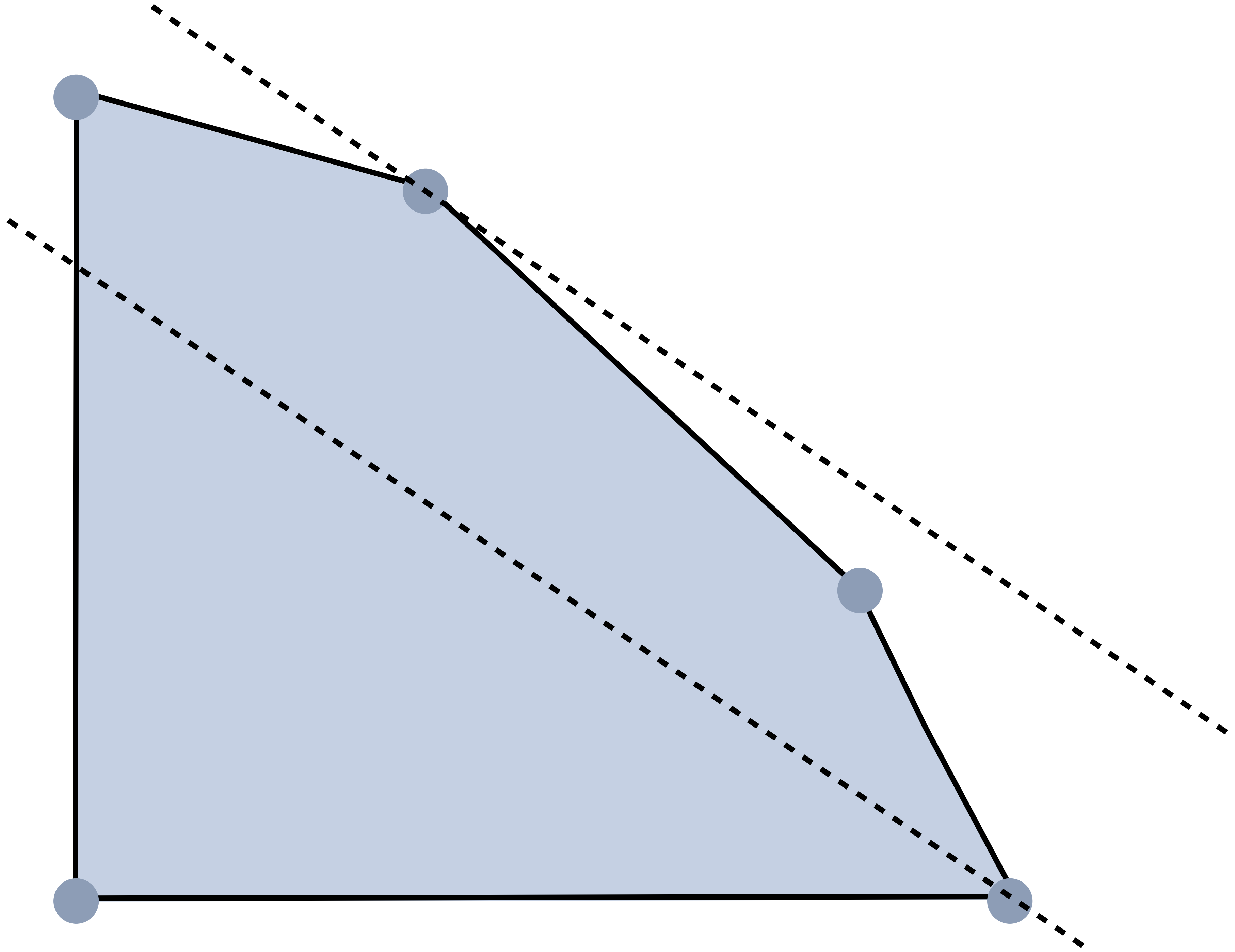
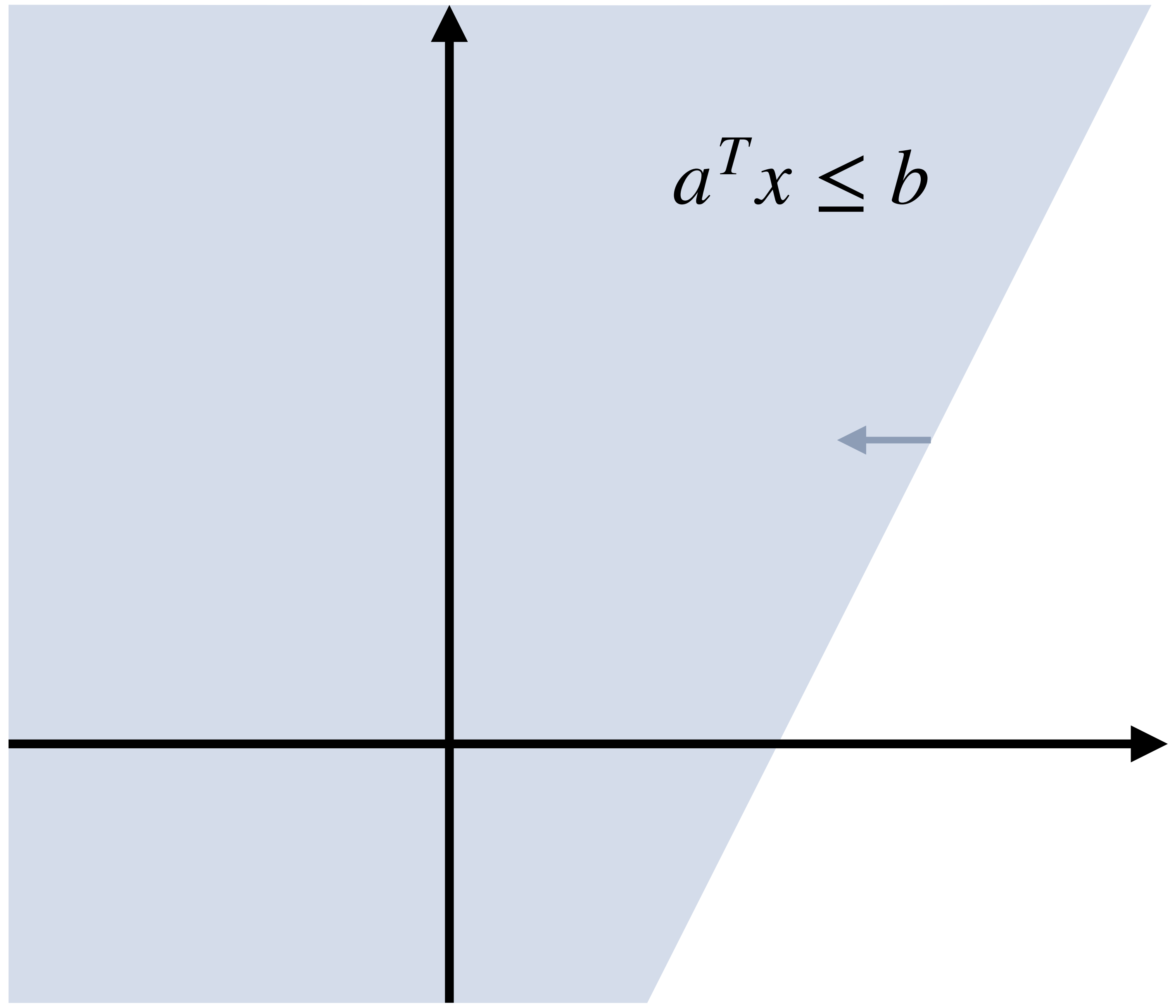




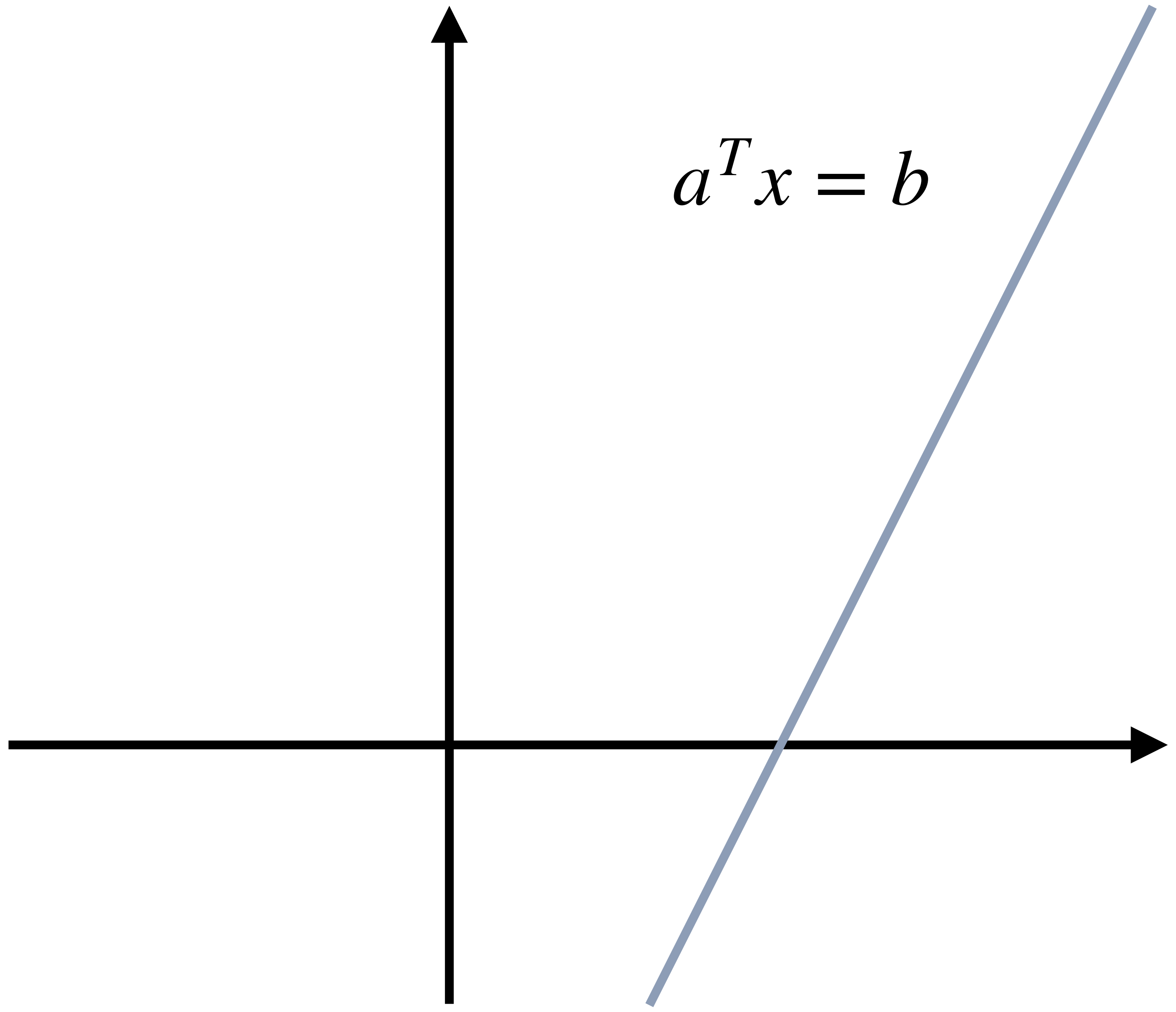
2 Geometry: Polyhedral Combinatorics

Halfspace. A halfspace in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n : a^T x \leq b\}$ for some vector $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Hyperplane. A hyperplane in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n : a^T x = b\}$ for some vector $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.



Halfspace

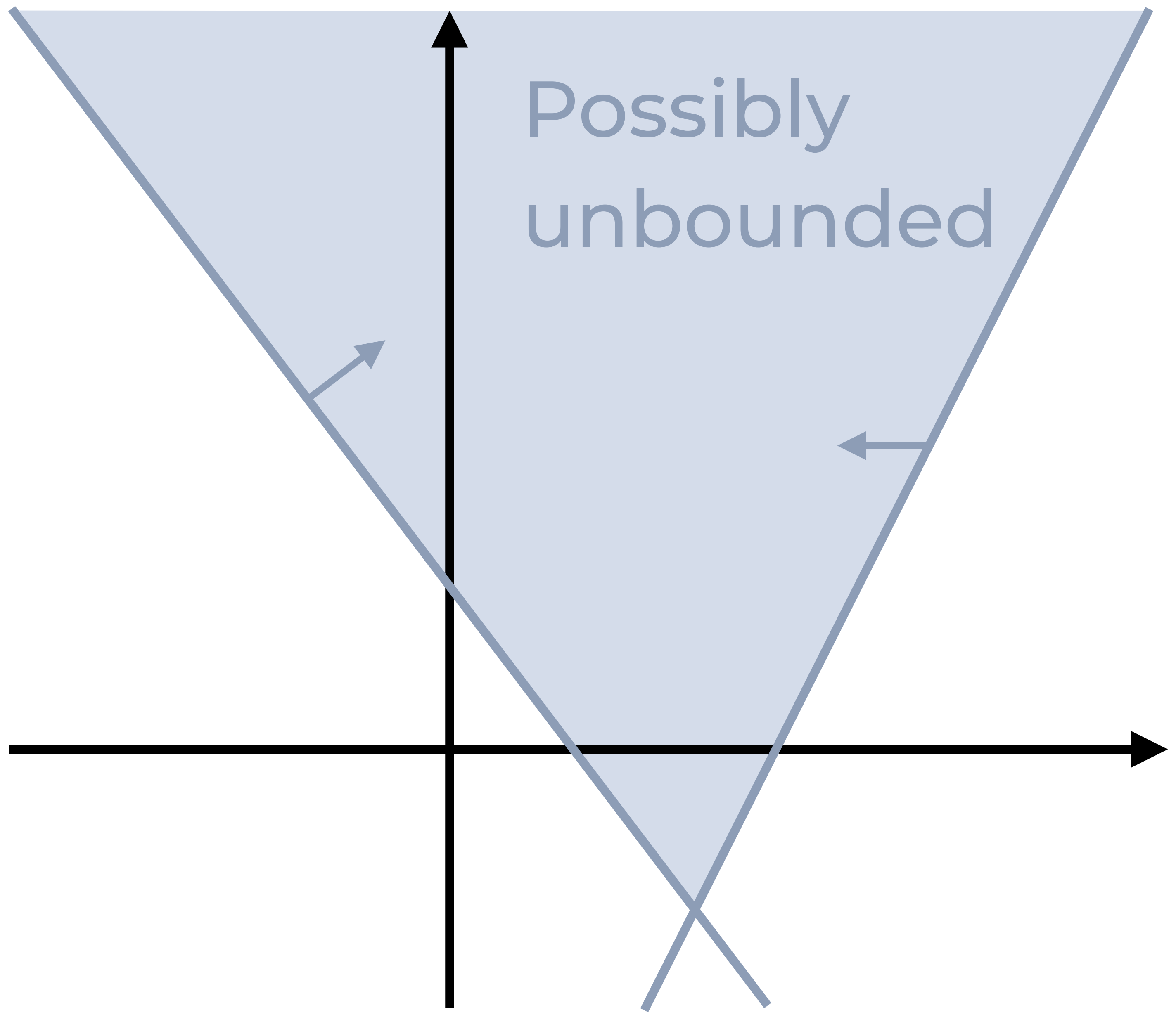


Hyperplane

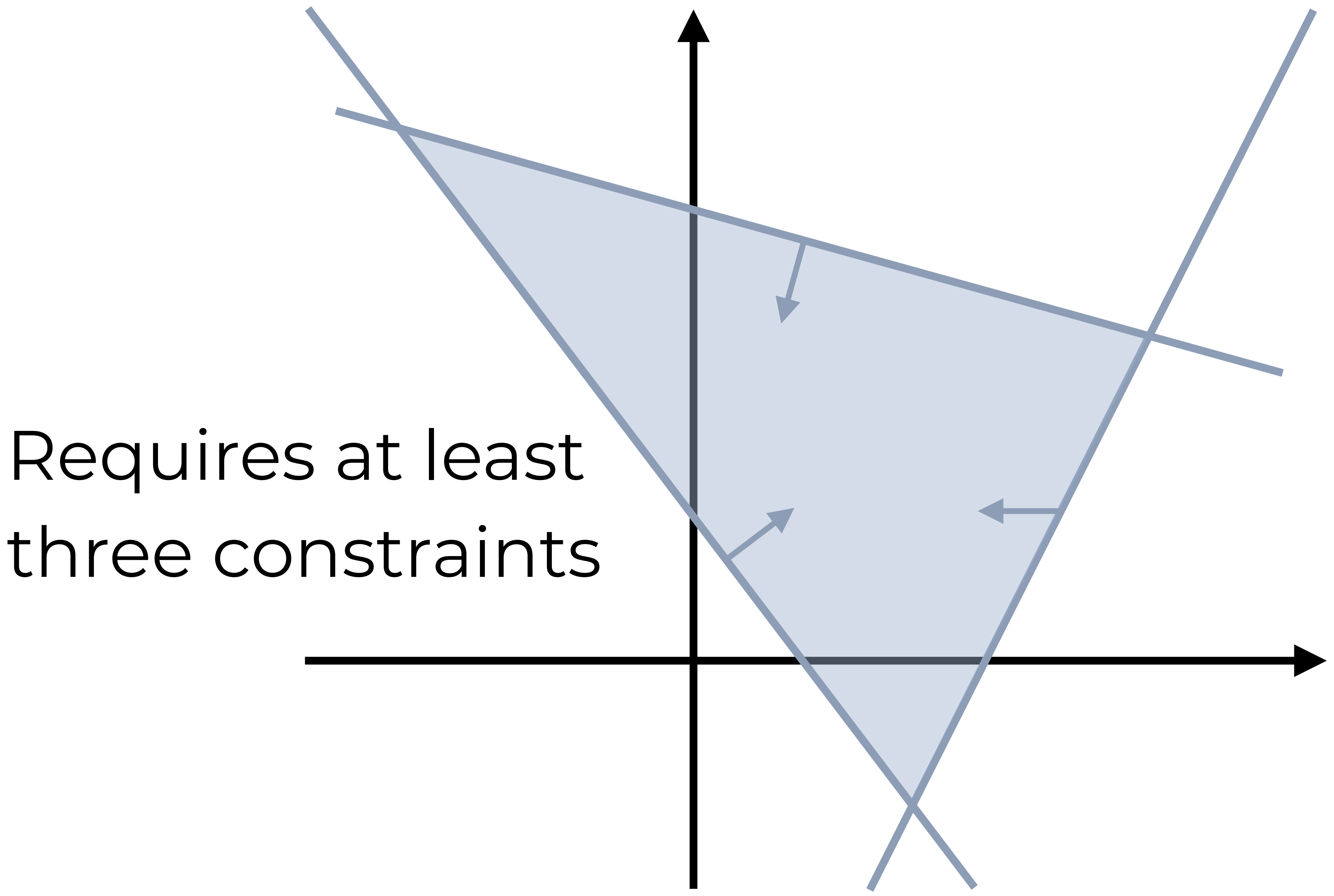
2 Geometry: Polyhedral Combinatorics

Polyhedron. A polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is the intersection of finitely many halfspaces.

Polytope. A polytope is a bounded polyhedron.



Polyhedron

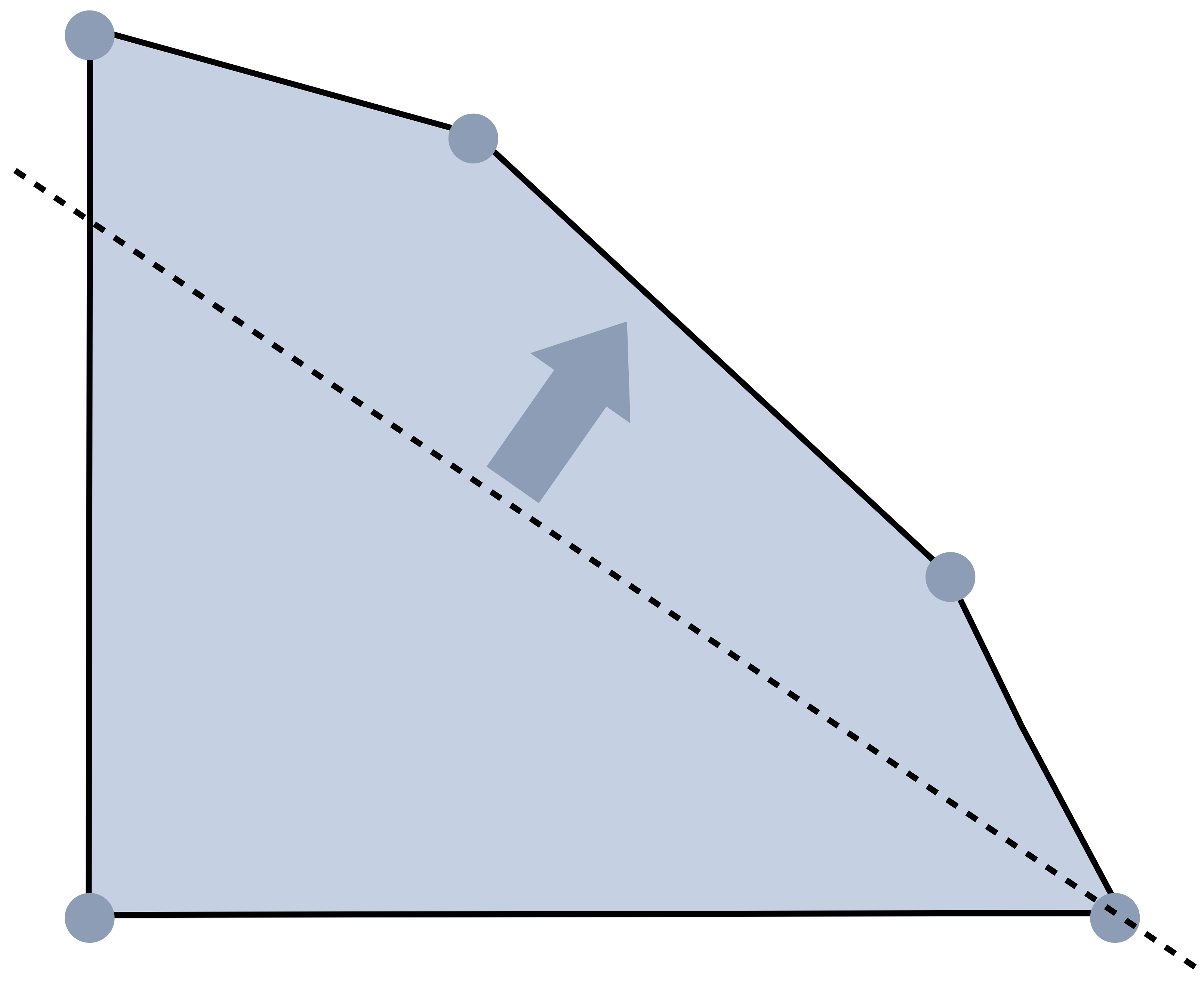


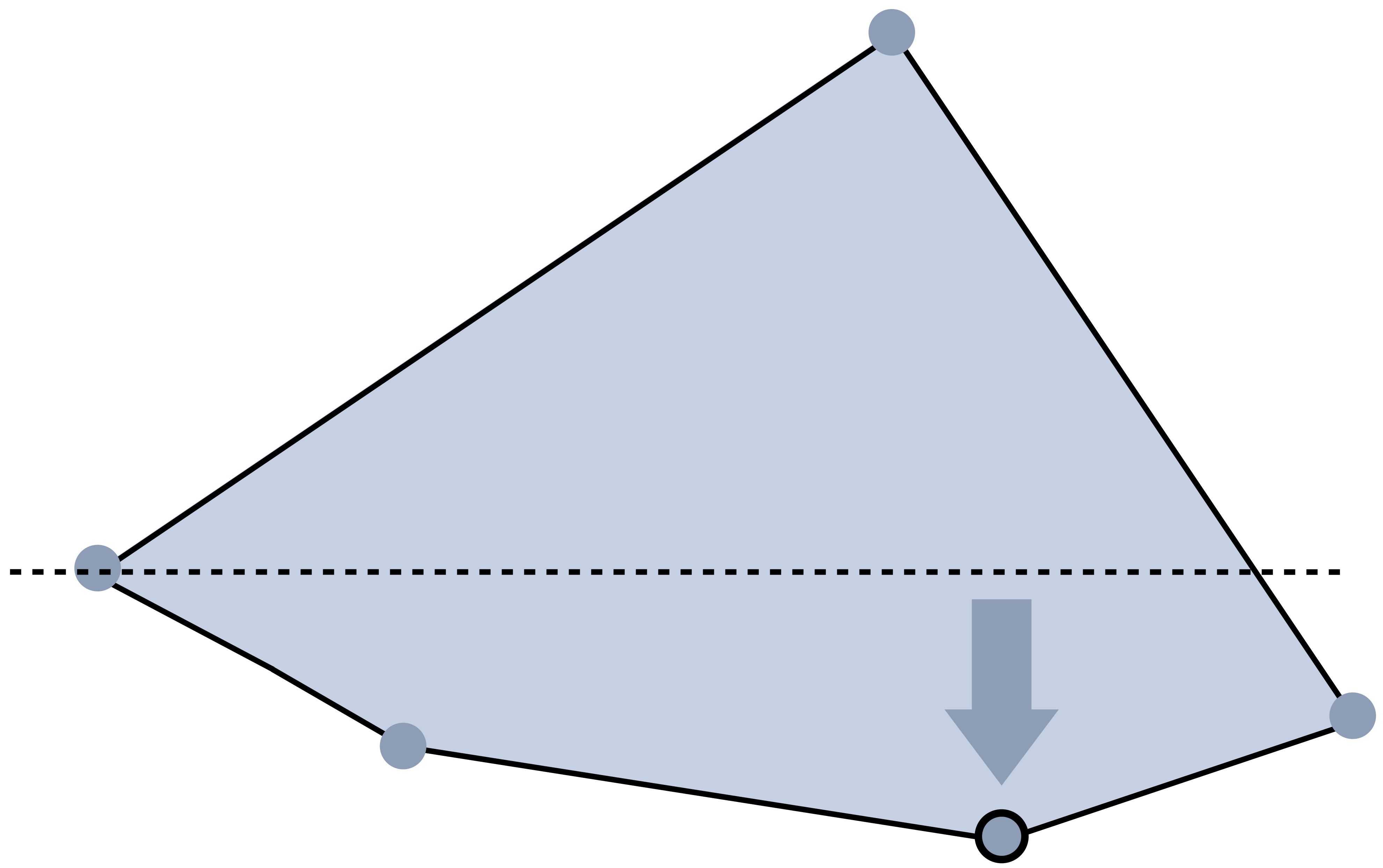
Polytope

2 Geometry: Polyhedral Combinatorics

By rotating $\mathbb{R}^n$ such that the objective function points downward, any LP can be expressed in the following geometric form:

Find the lowest point in a given polyhedron.





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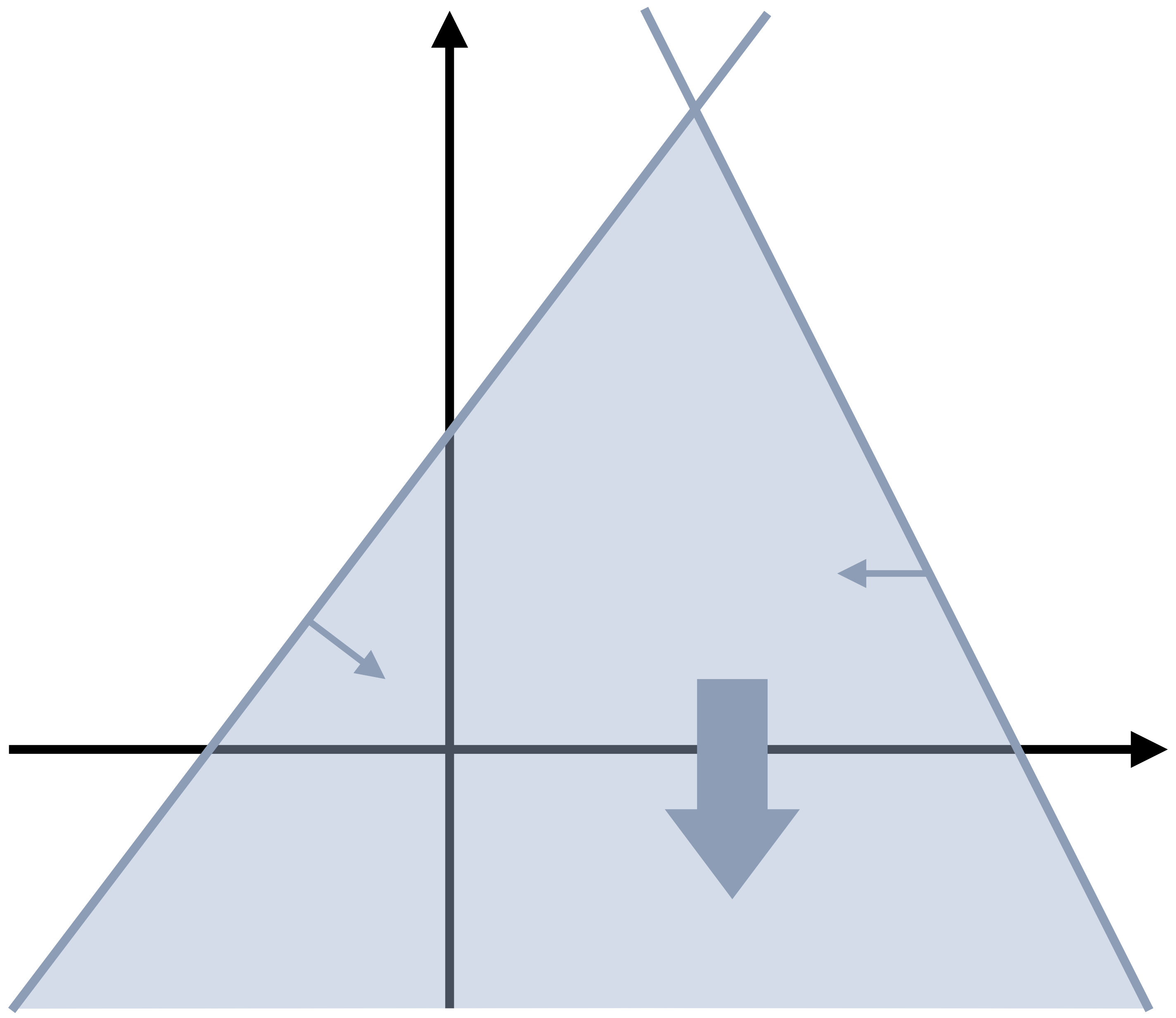
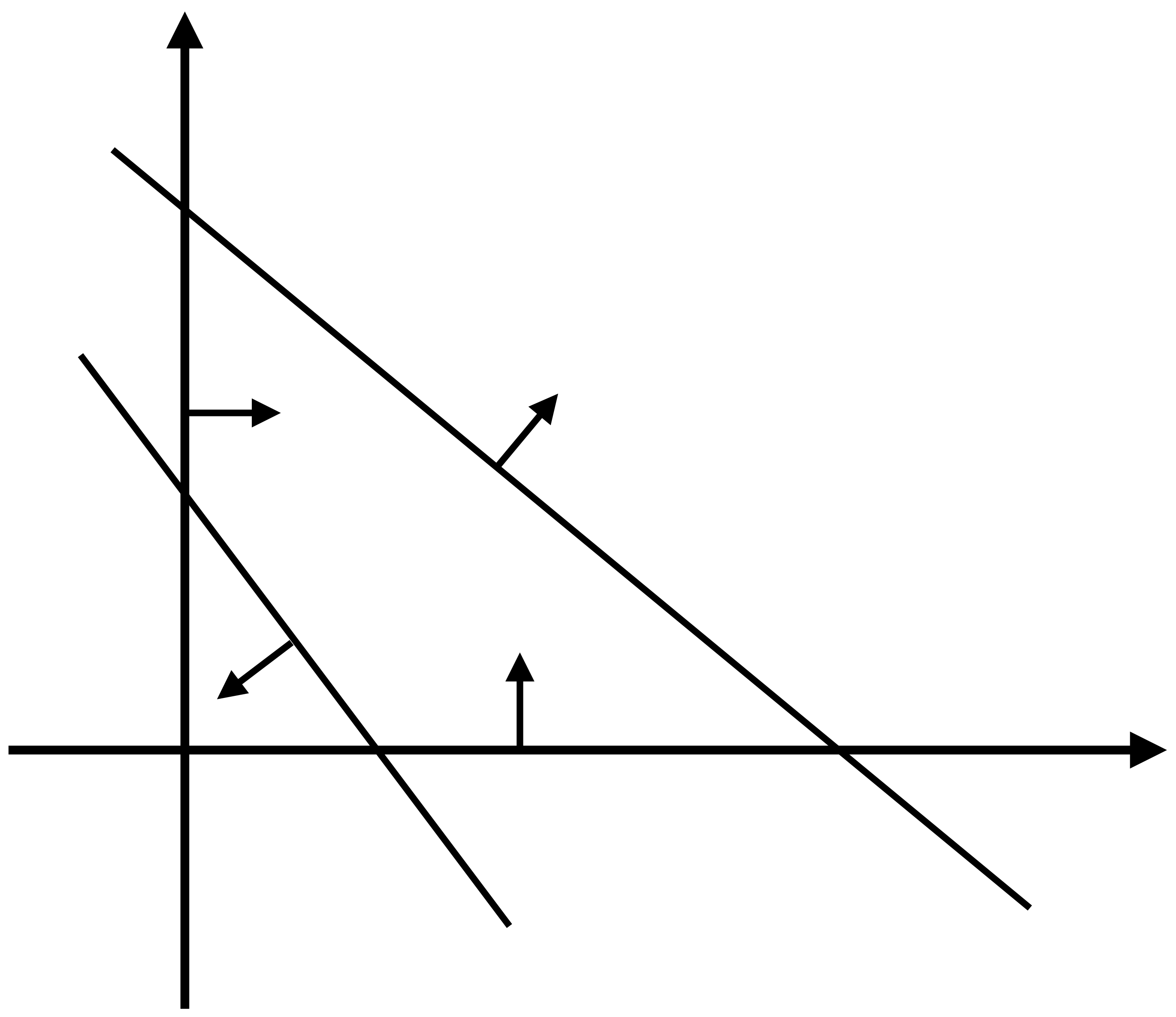


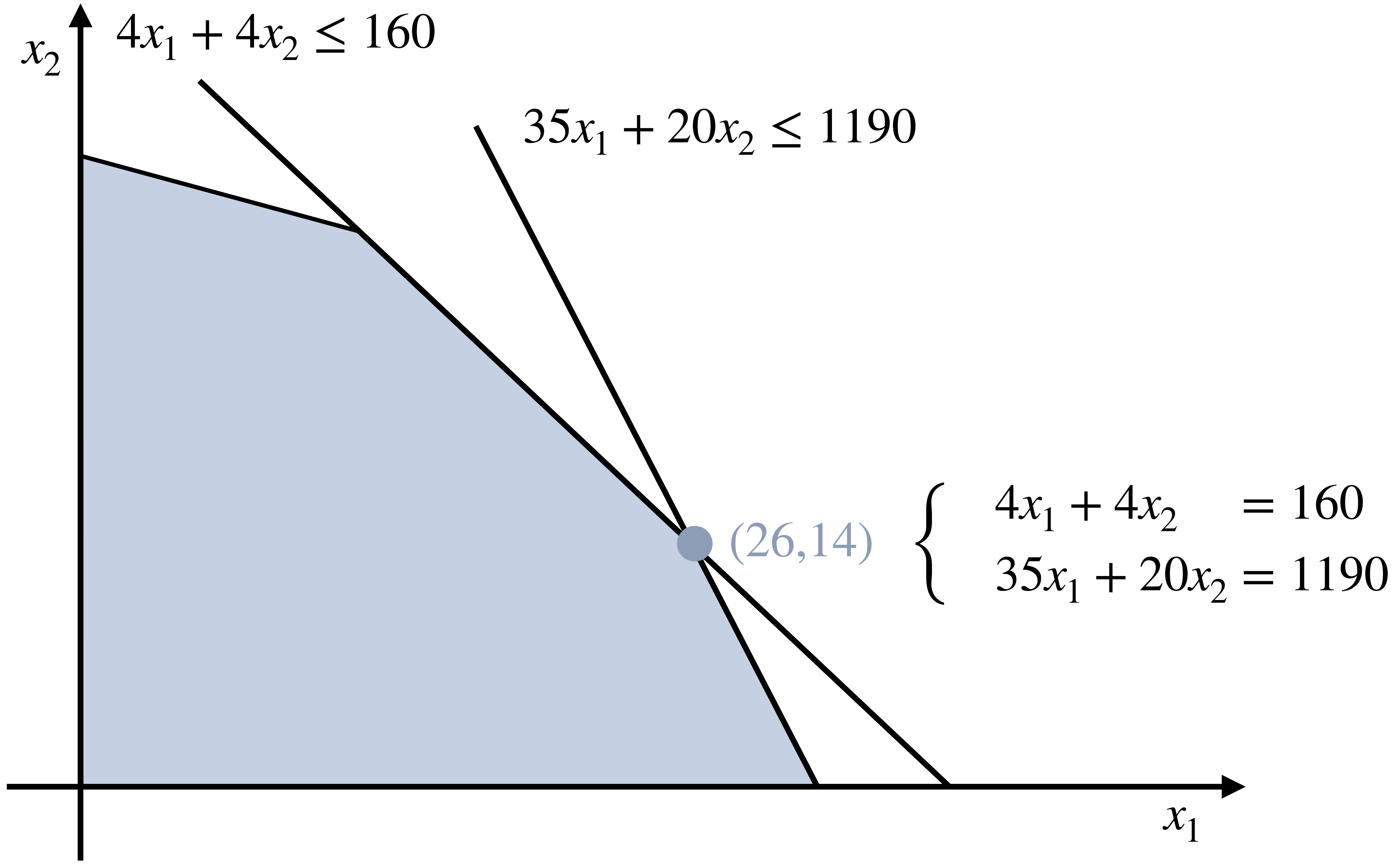
Unbounded



Infeasible

Geometry: Polyhedral Combinatorics

Intuition Vertex. A vertex in $\mathbb{R}^n$ is uniquely specified by n linearly independent equations.



$$\begin{cases} 4x_1 + 4x_2 = 160 \\ 35x_1 + 20x_2 = 1190 \end{cases}$$

2 Geometry: Polyhedral Combinatorics

Theorem. Given $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$, $x \in P$ is a basic feasible solution (BFS) iff there exists a basis $B \subseteq \{1,\ldots,n\}$ such that $|B| = m$ and

- $A_B \in \mathbb{R}^{m \times m}$ is nonsingular,
- $x_B = A_B^{-1}b \geq 0$,
- $x_N = \mathbf{0}$.

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- $A_B \in \mathbb{R}^{m\times m}$ is nonsingular,
- $x_B = A_B^{-1} b \geq 0$,
- $x_N = \mathbf{0}$.

Notation.

- Let B be the set of column indices, then A_B is the submatrix of A indexed by B

$$\begin{array}{rcccc} B = & \{1, & & 3, & 4\} \\ & 2 & 1 & 3 & 0 \\ A = & 7 & 3 & 2 & 1 \\ & 0 & 0 & 0 & 5 \end{array}$$

$$A_B$$

2 Geometry: Polyhedral Combinatorics

Theorem. Given $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$, $x \in P$ is a basic feasible solution (BFS) iff there exists a basis $B \subseteq \{1,\ldots,n\}$ such that $|B| = m$ and

- $A_B \in \mathbb{R}^{m \times m}$ is nonsingular,
- $x_B = A_B^{-1} b \geq 0$,
- $x_N = \mathbf{0}$.

Notation.

- Let B be the set of column indices, then A_B is the submatrix of A indexed by B
- Let x_B denote the m components of x associated with A_B

x_B

$$x^T = \begin{pmatrix} 2 & 0 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

A_B

2 Geometry: Polyhedral Combinatorics

Theorem. Given $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$, $x \in P$ is a basic feasible solution (BFS) iff there exists a basis $B \subseteq \{1, \ldots, n\}$ such that $|B| = m$ and

- $A_B \in \mathbb{R}^{m \times m}$ is nonsingular,
- $x_B = A_B^{-1} b \geq 0$,
- $x_N = \mathbf{0}$.

Notation.

- Let B be the set of column indices, then A_B is the submatrix of A indexed by B
- Let x_B denote the m components of x associated with A_B
- Let denote x_N the $n - m$ components of x not associated with A_B

$$\begin{array}{cccccc} & & & x_N & & \\ x^T = & (& 2 & 0 & 1 & 0 \ ) \end{array}$$

$$A = \begin{array}{c} \begin{pmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix} \\ A_B \end{array}$$

2 Geometry: Polyhedral Combinatorics

Theorem. Given $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$, $x \in P$ is a basic feasible solution (BFS) iff there exists a basis $B \subseteq \{1,\ldots,n\}$ such that $|B| = m$ and

- $A_B \in \mathbb{R}^{m \times m}$ is nonsingular,
- $x_B = A_B^{-1} b \geq 0$,
- $x_N = \mathbf{0}$.

Example.

$$A = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad x = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 16 \\ 0 \end{pmatrix}, \quad B = \{1,3,4\}, \quad N = \{2\}$$

$$A_B x_B = \begin{pmatrix} 2 & 3 & 0 \\ 7 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 16 \\ 0 \end{pmatrix} = b$$

2 Geometry: Polyhedral Combinatorics

Theorem.

i) If there exists a feasible solution, there exists a basic feasible solution.

ii) If there exists an optimal feasible solution, there exists an optimal basic feasible solution.

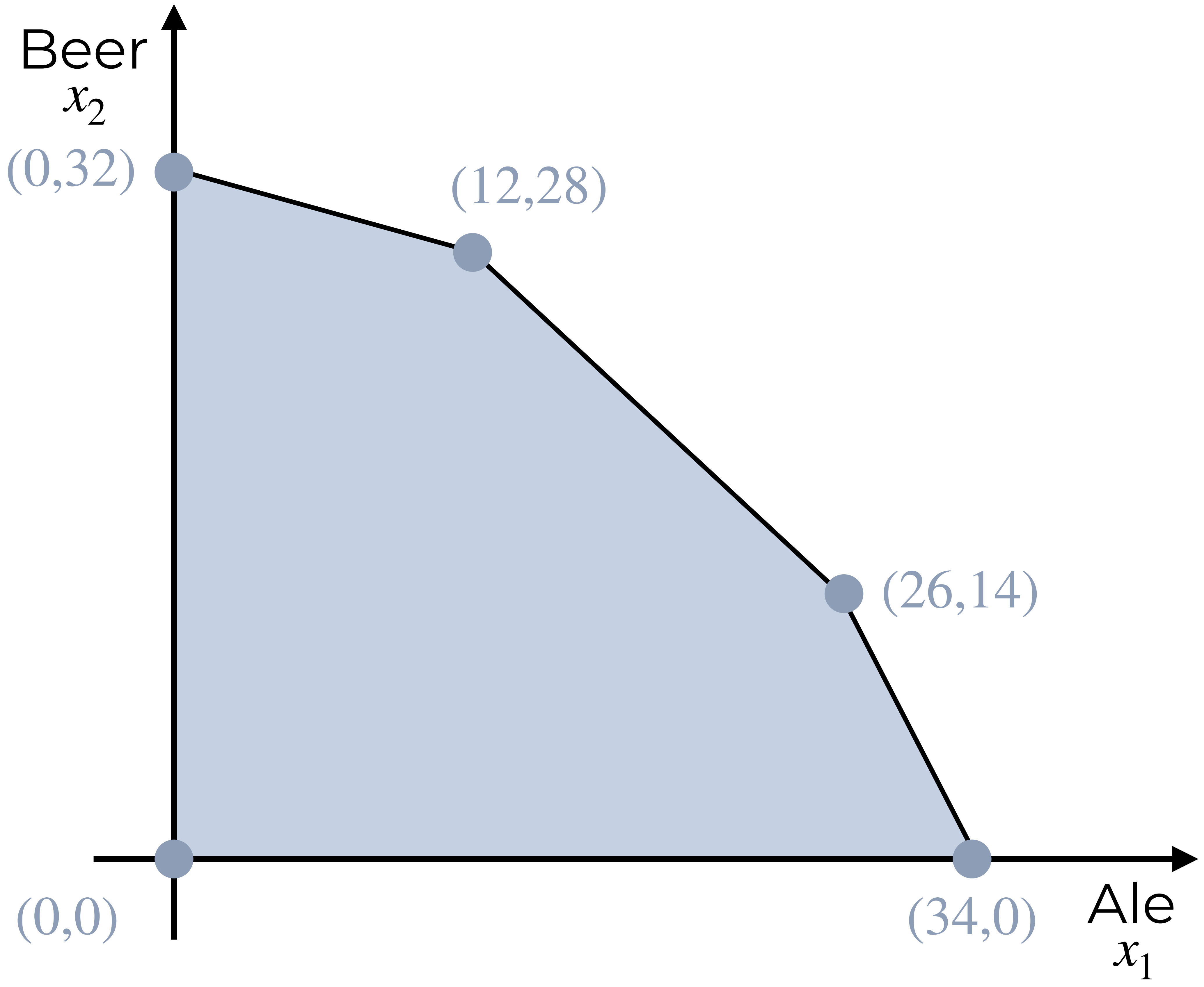
Observation. Thus, the task of solving a LP is reduced to that of searching over basic feasible solutions. For a problem with n variables and m constraints there are at most

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

basic feasible solutions.

2 Brewery Example: Basic Feasible Solutions

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 5 & 15 & 1 & & \\ 4 & 4 & & 1 & \\ 35 & 20 & & & 1 \end{pmatrix} \end{array} \cdot x = \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$



2 Brewery Example: Basic Feasible Solutions

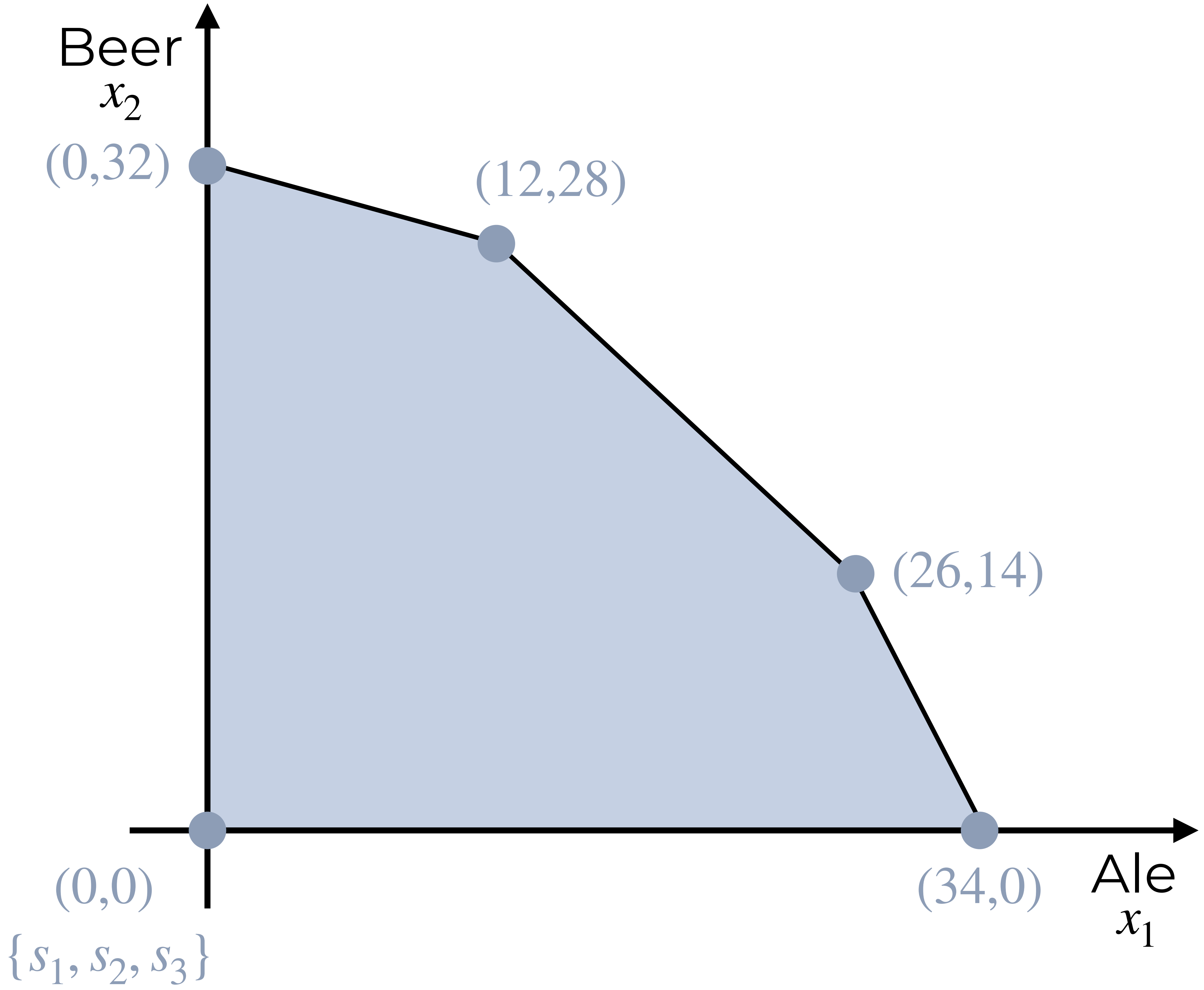
$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \end{array}$$

$$\begin{pmatrix} 5 & 15 & 1 & & \\ 4 & 4 & & 1 & \\ 35 & 20 & & & 1 \end{pmatrix} \cdot x = \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$

$$\Downarrow$$

$$x_1 = 0$$

$$x_2 = 0$$

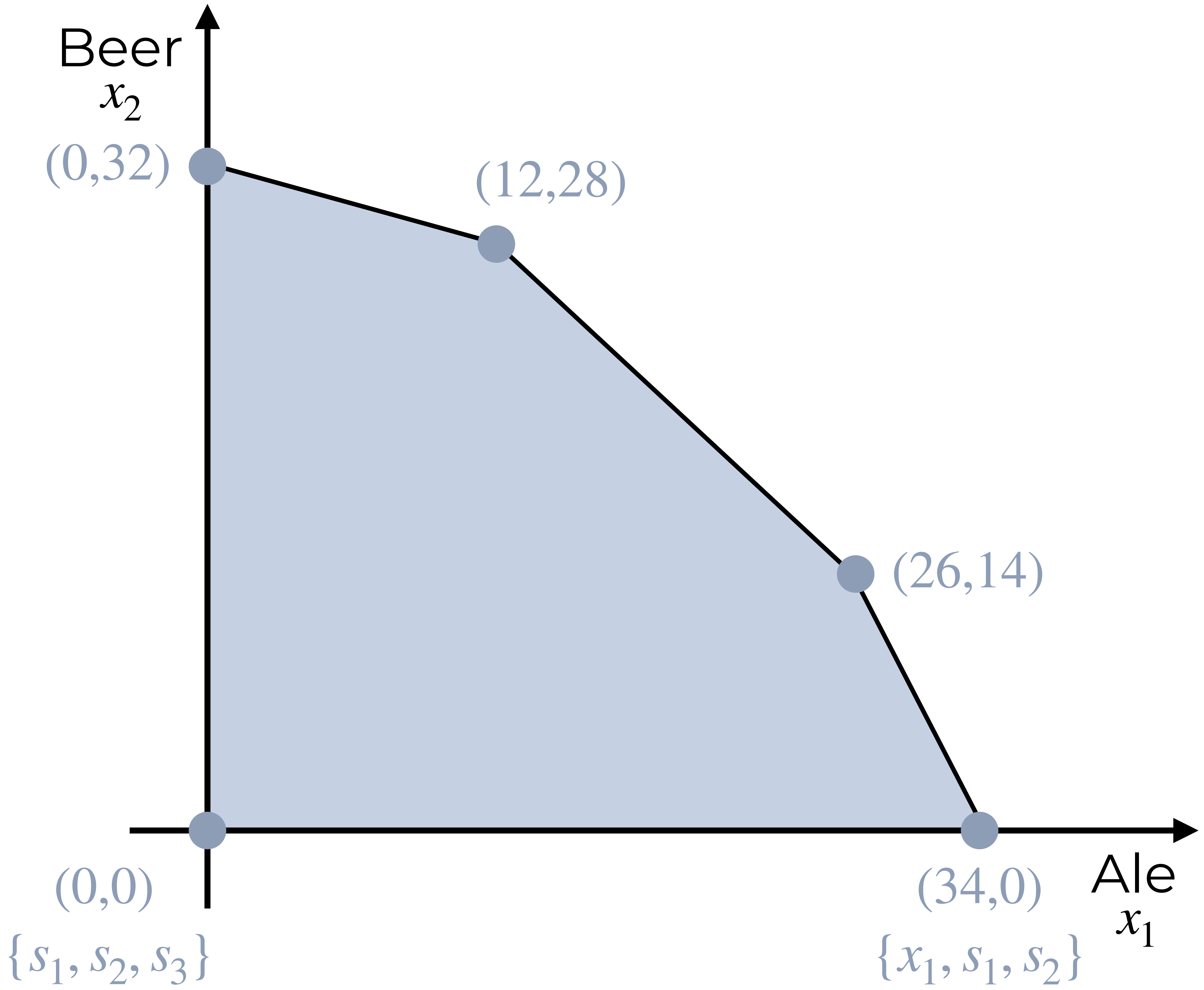


2 Brewery Example: Basic Feasible Solutions

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \\ 5 & 15 & 1 & & \\ 4 & 4 & & 1 & \\ 35 & 20 & & & 1 \end{array} \cdot x = \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$

$$\Downarrow$$

$$x_1 = 34$$
$$x_2 = 0$$



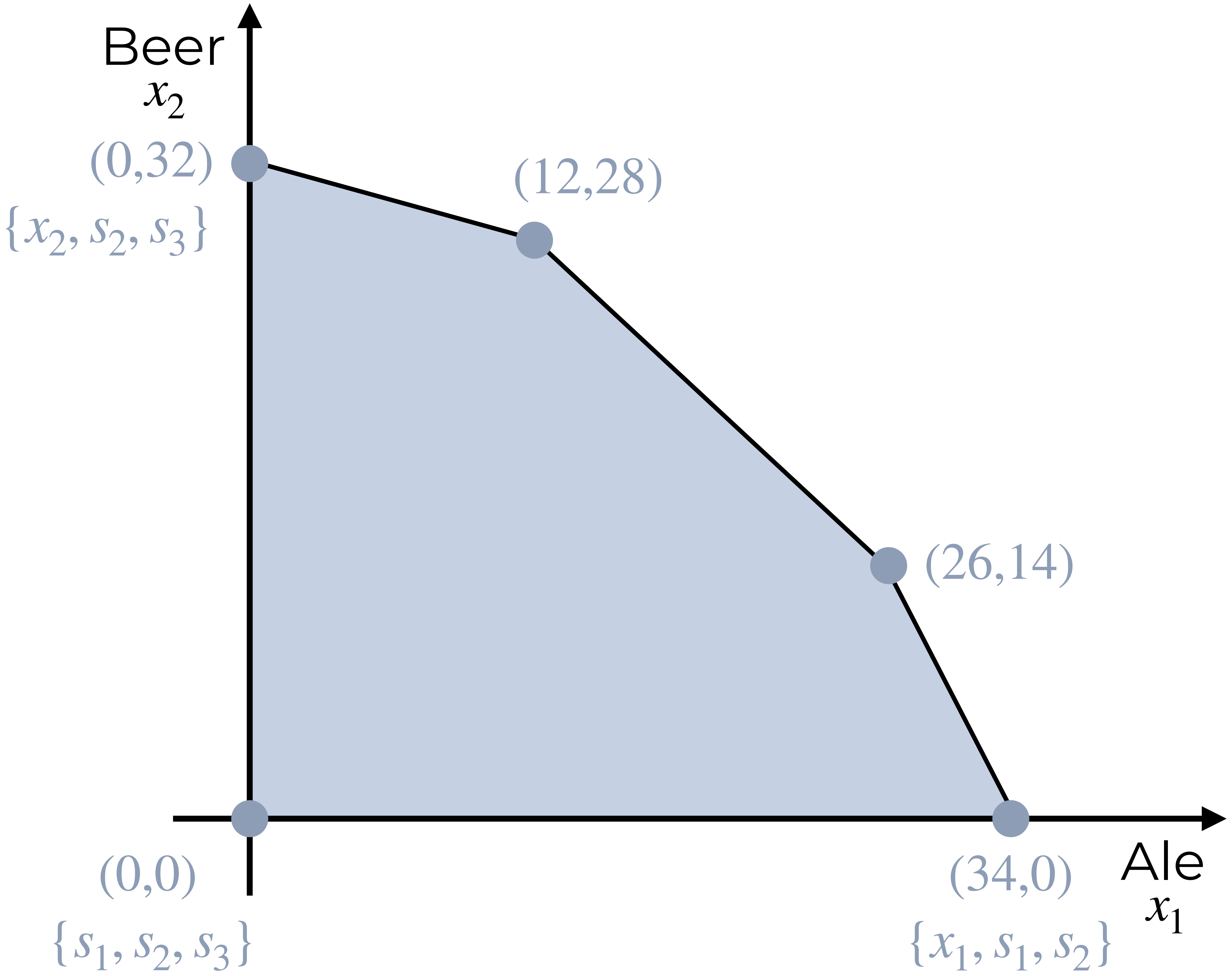
2 Brewery Example: Basic Feasible Solutions

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \end{array}$$

$$\begin{pmatrix} 5 & 15 & 1 & & \\ 4 & 4 & & 1 & \\ 35 & 20 & & & 1 \end{pmatrix} \cdot x = \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$

$$\Downarrow$$

$$x_1 = 0$$
$$x_2 = 32$$

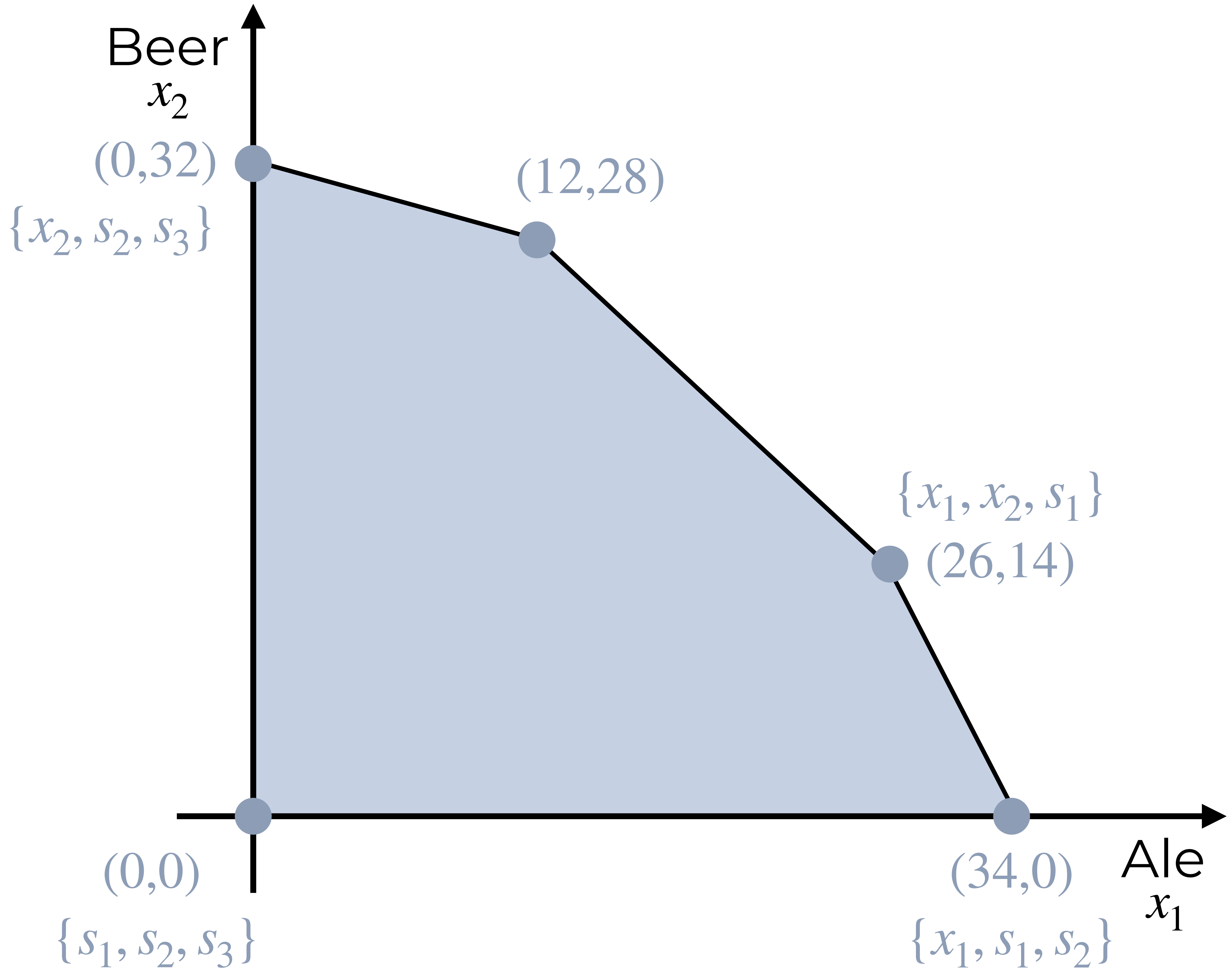


2 Brewery Example: Basic Feasible Solutions

$$\left(\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \\ 5 & 15 & 1 & & \\ 4 & 4 & & 1 & \\ 35 & 20 & & & 1 \end{array}\right) \cdot x = \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$

$$\Downarrow$$

$$x_1 = 26$$
$$x_2 = 14$$



2 Brewery Example: Basic Feasible Solutions

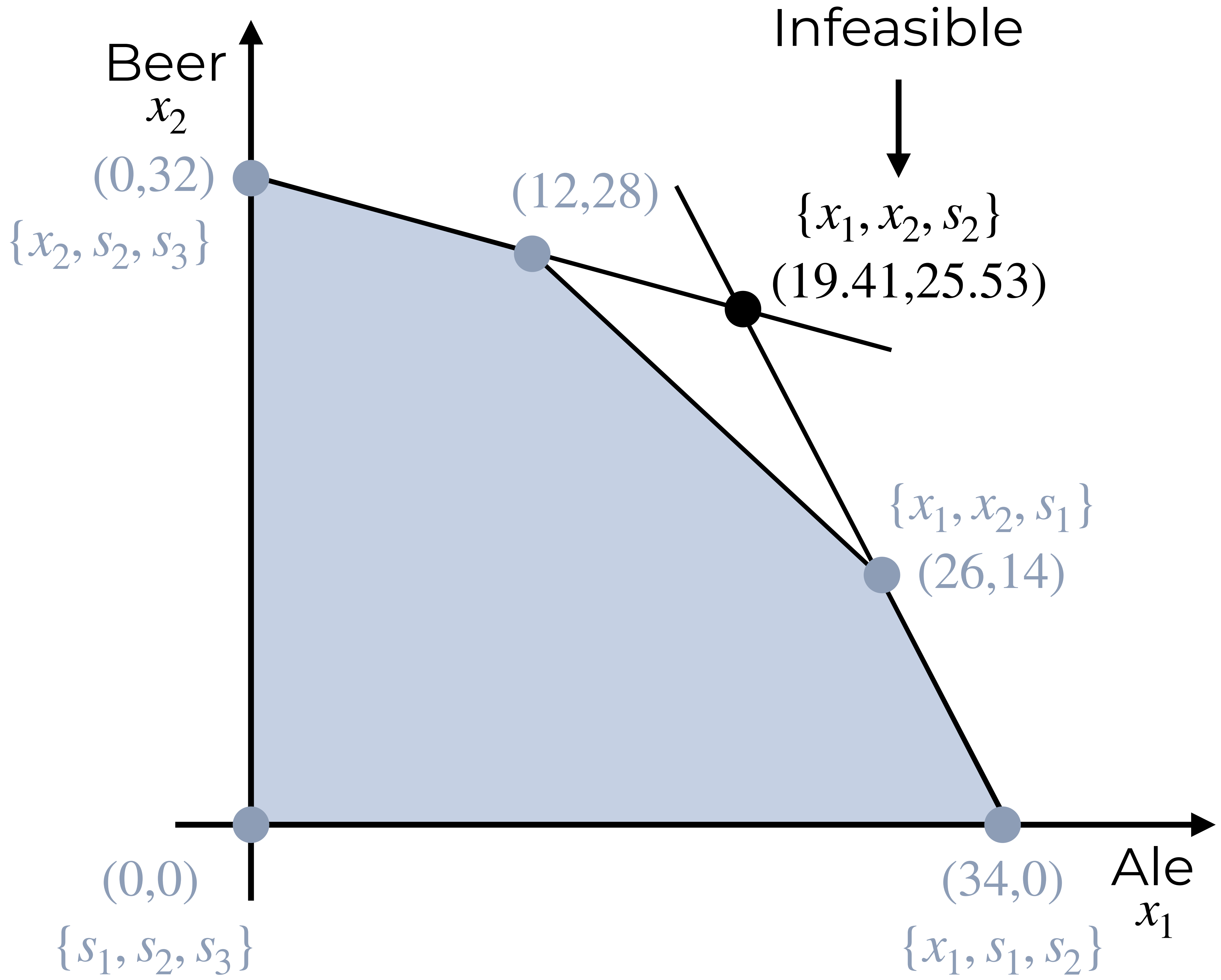
$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \end{array}$$

$$\begin{pmatrix} 5 & 15 & 1 & & \\ 4 & 4 & & 1 & \\ 35 & 20 & & & 1 \end{pmatrix} \cdot x = \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$

$$\Downarrow$$

$$x_1 = 19.41$$

$$x_2 = 25.53$$



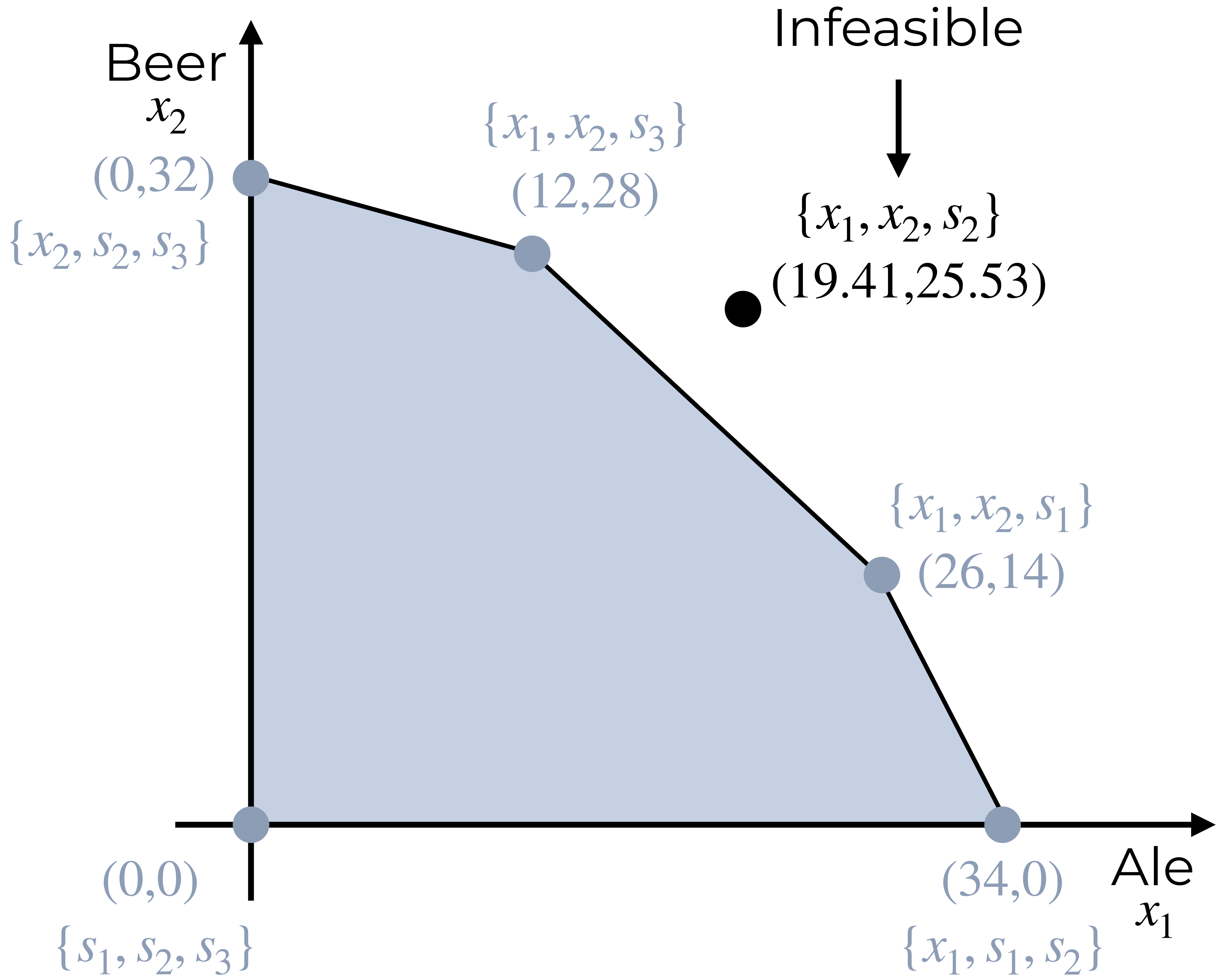
2 Brewery Example: Basic Feasible Solutions

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \\ \end{array}$$

$$\begin{pmatrix} 5 & 15 & 1 & & \\ 4 & 4 & & 1 & \\ 35 & 20 & & & 1 \end{pmatrix} \cdot x = \begin{pmatrix} 480 \\ 160 \\ 1190 \end{pmatrix}$$

$$\Downarrow$$

$$x_1 = 12$$
$$x_2 = 28$$

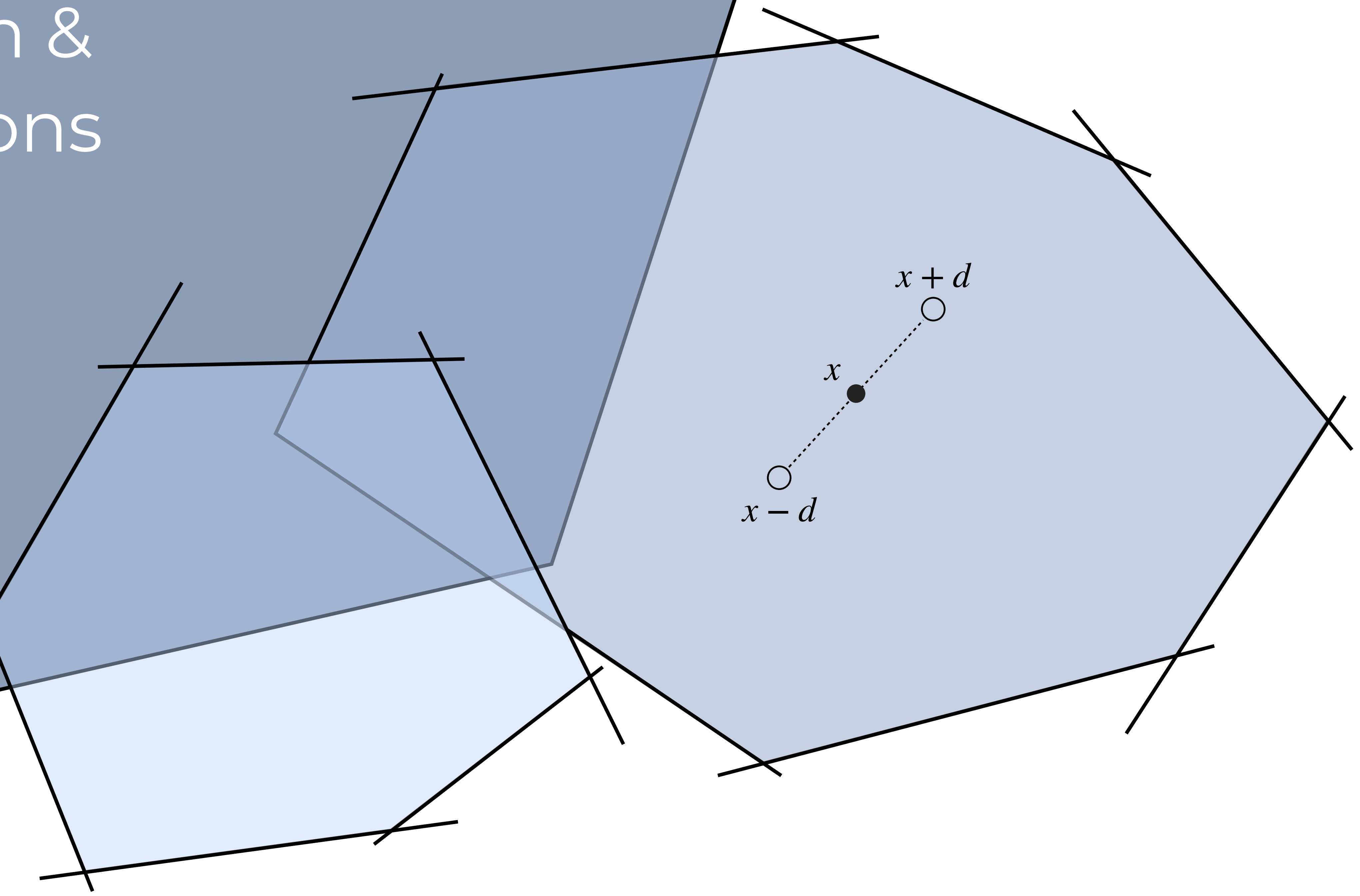


Linear Programming I

LP Definition

Feasible Region & Optimal Solutions

LP Algorithms



3 LP Algorithms

Goal. Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, solve $\max\{c^T x : Ax \leq b, x \in \mathbb{R}^n\}$.

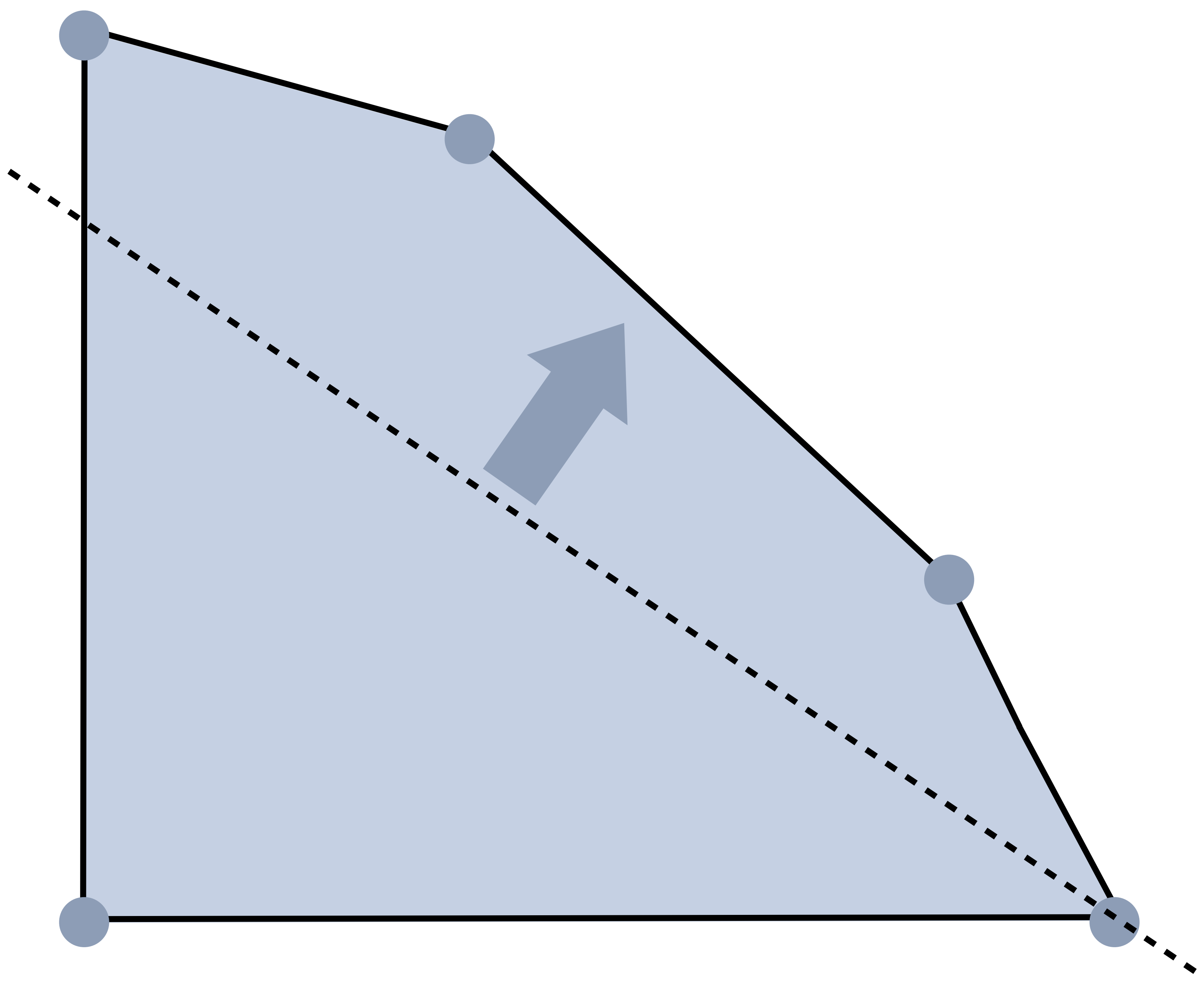
LP Algorithms.

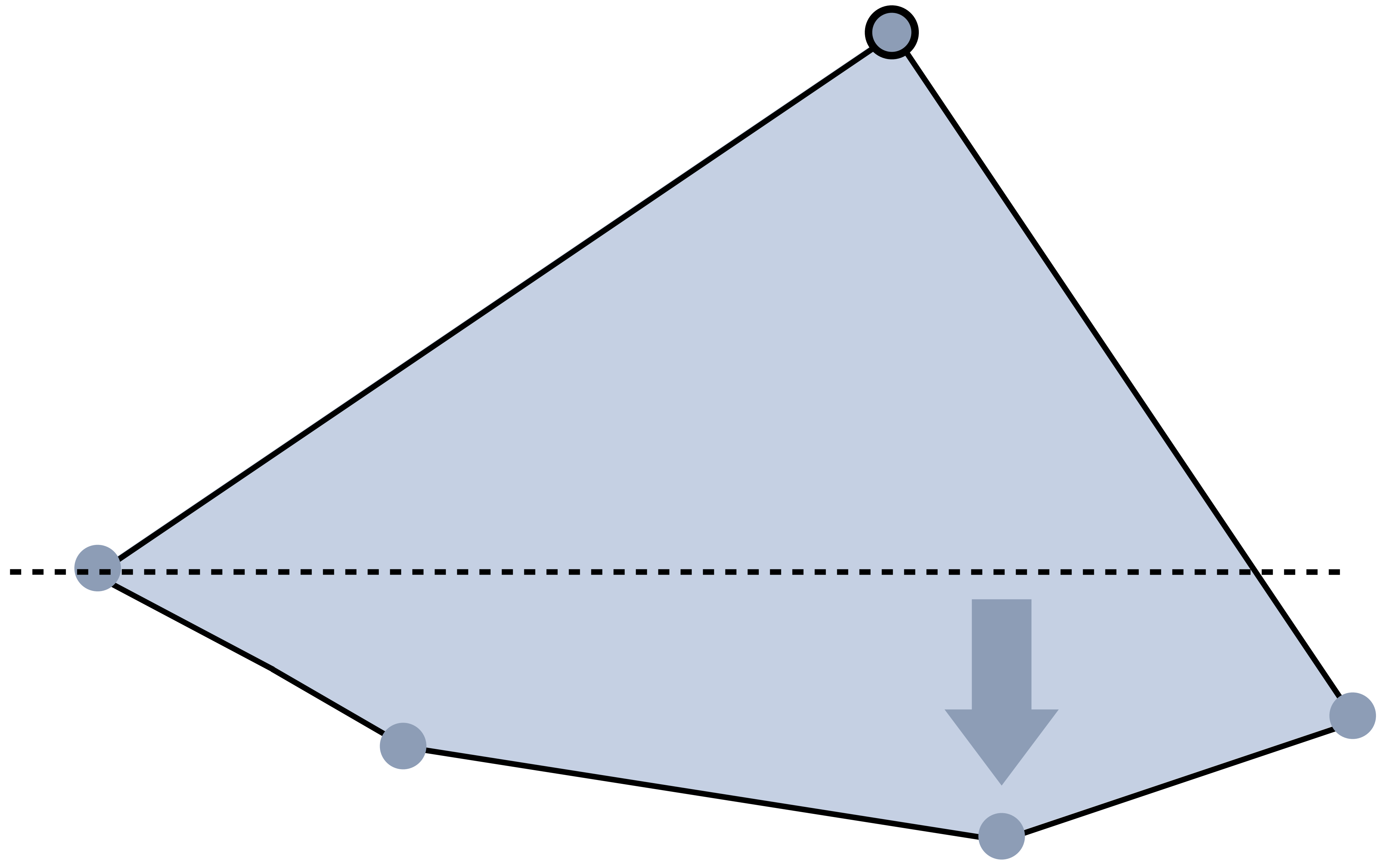
- Simplex Method. Search over basic feasible solutions.
- Ellipsoid Algorithm. Use geometric divide-and-conquer.

3 Simplex Method: Intuition

Input. Set of halfspaces H
Output. Lowest vertex in the intersection of halfspaces in H

Intuition. Follow a falling marble along the sides of the polyhedron.





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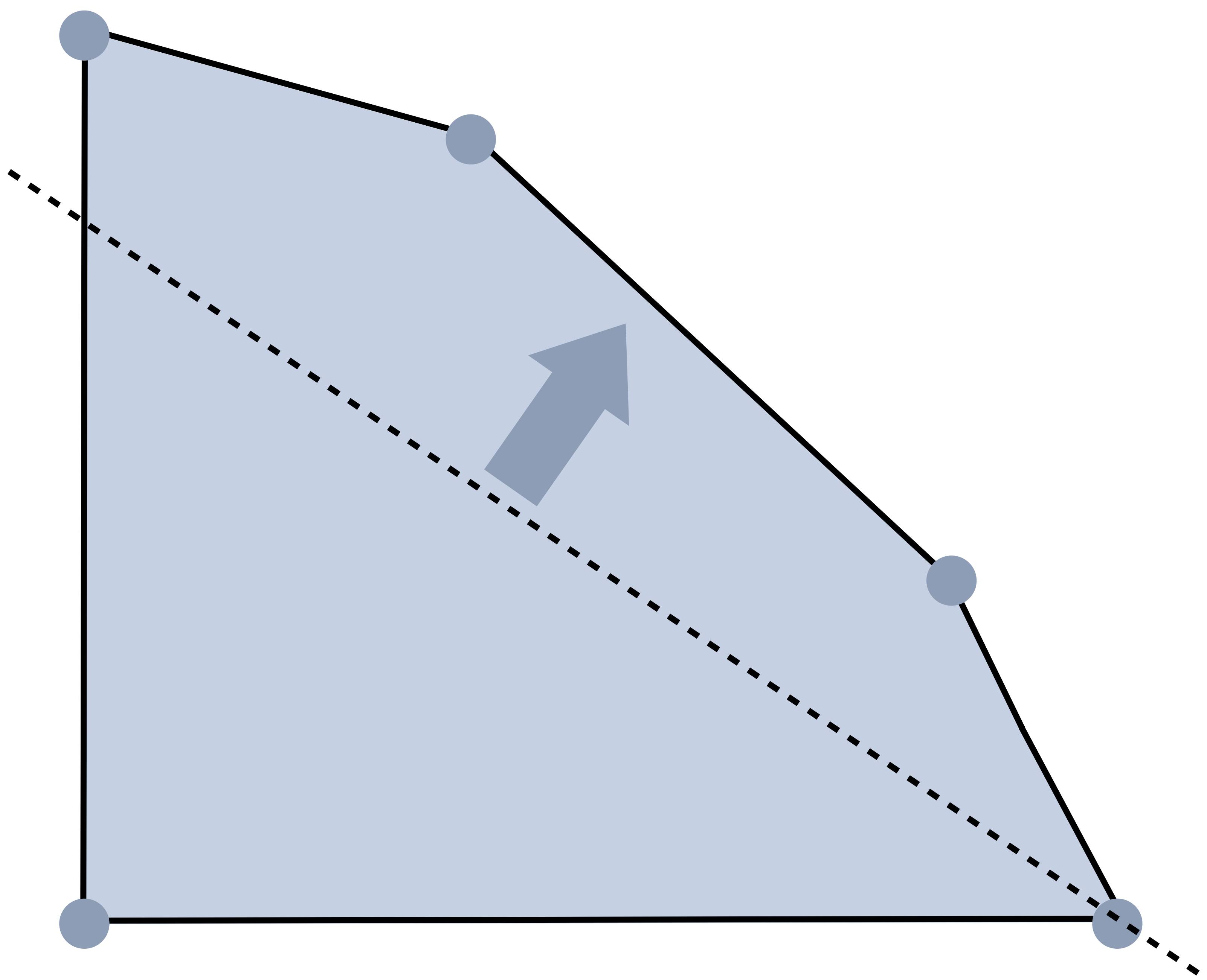

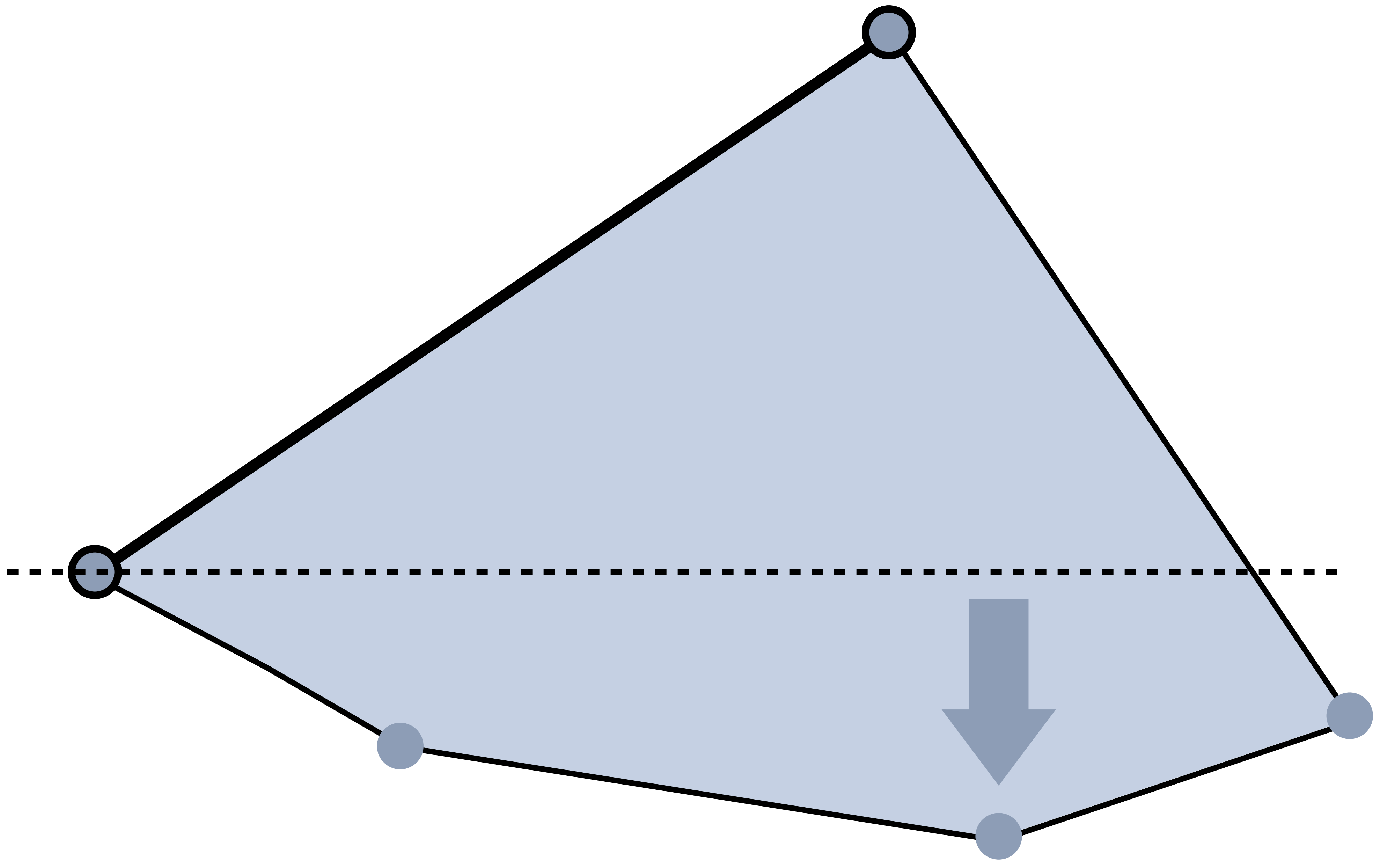

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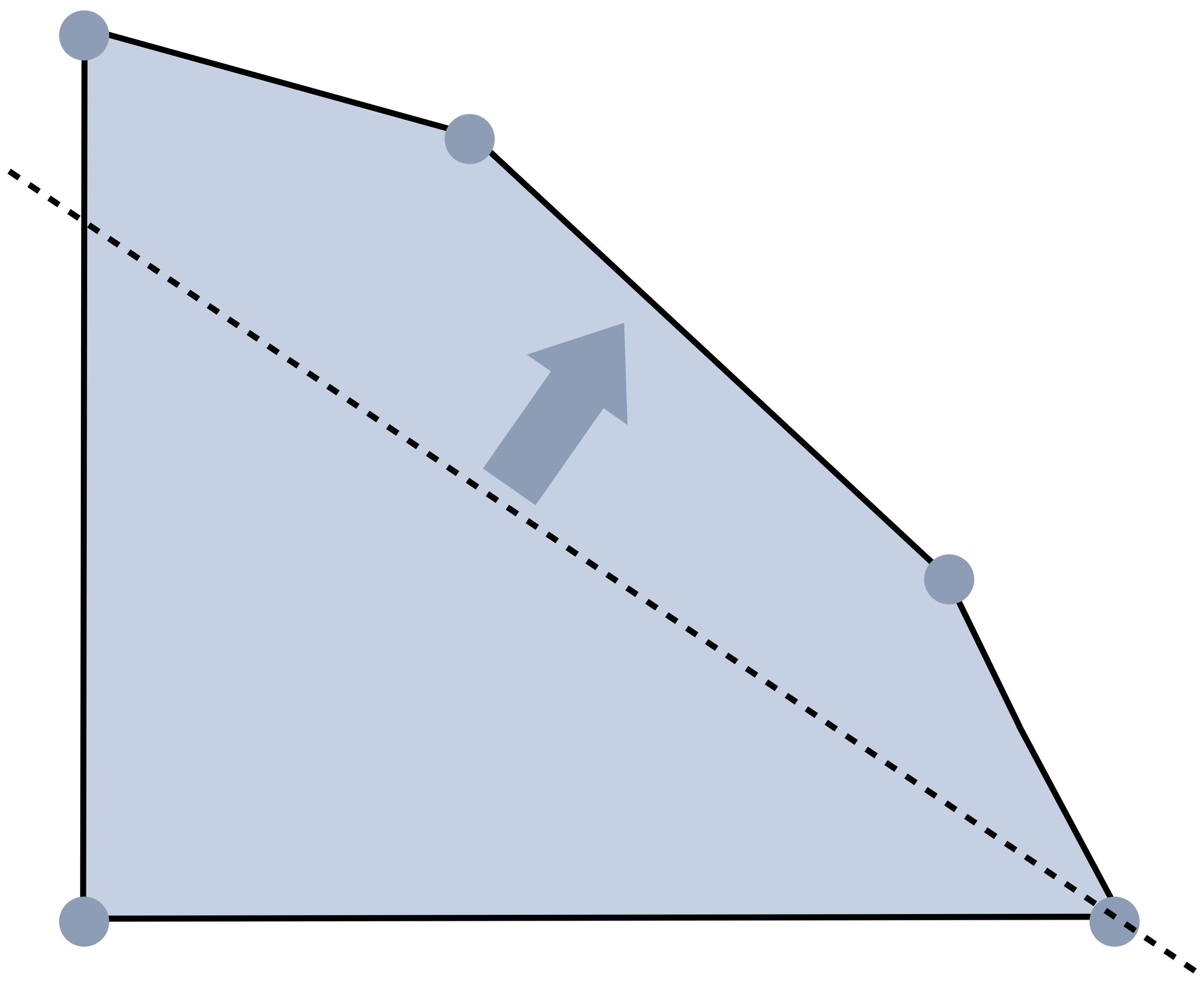

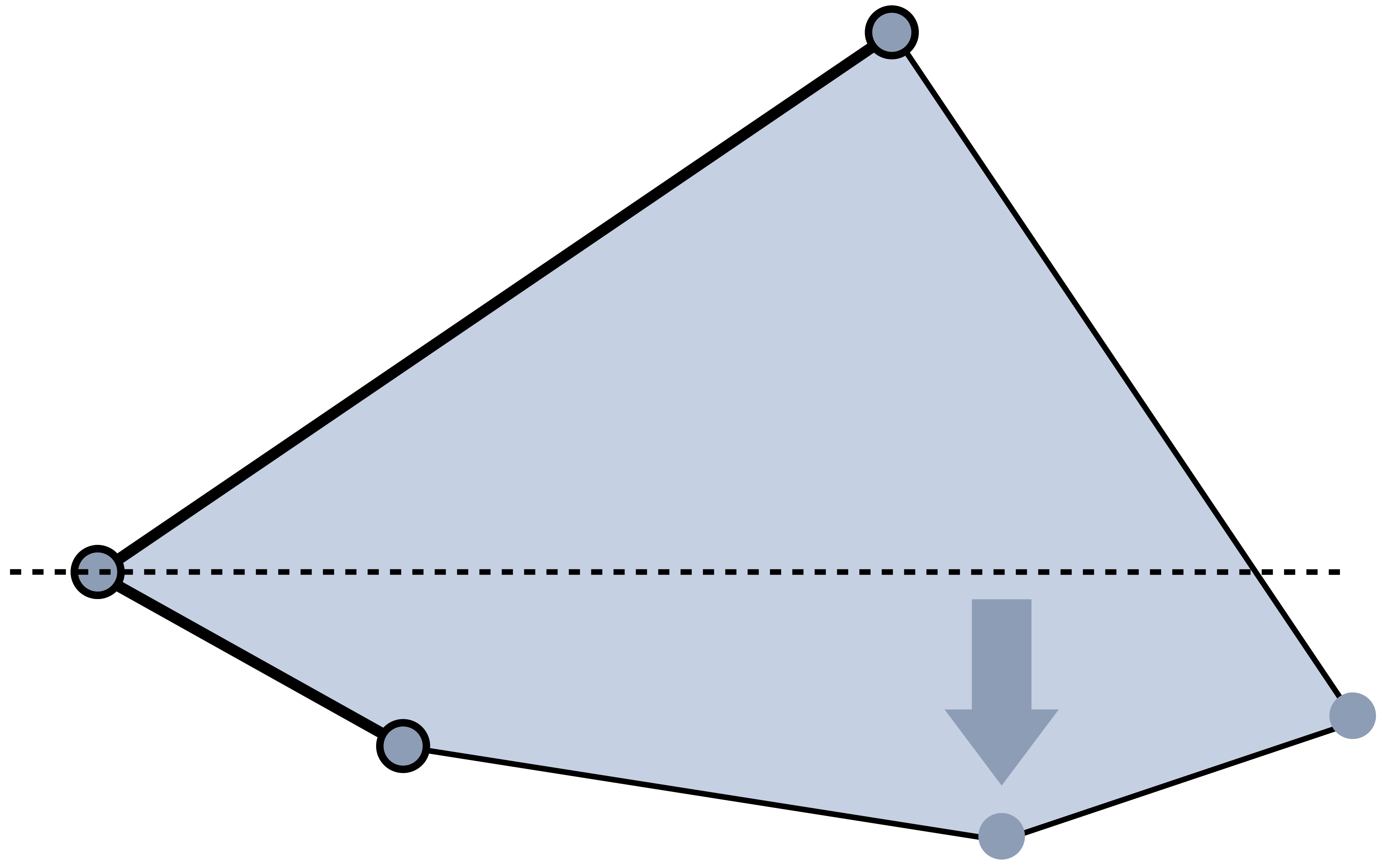

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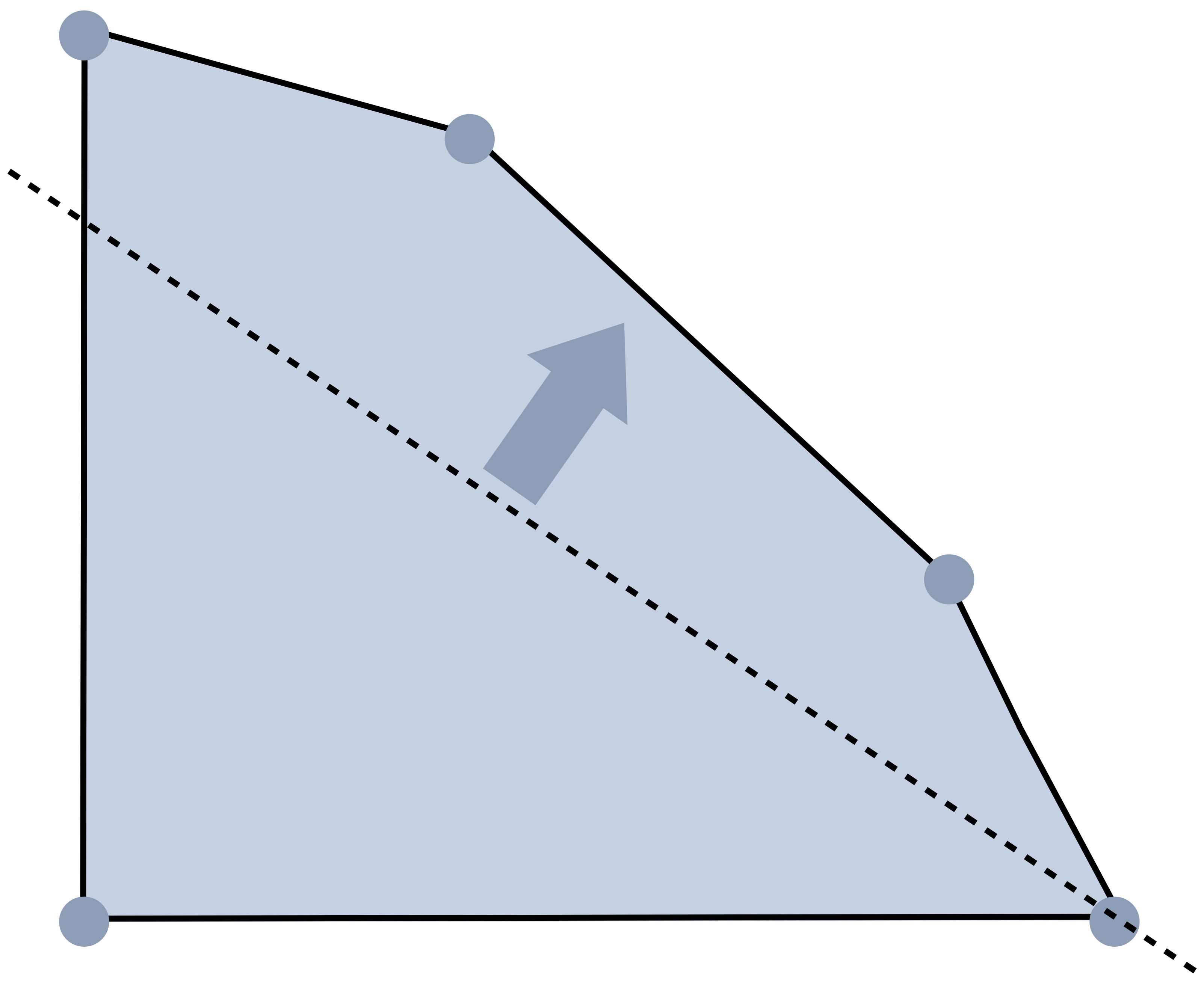
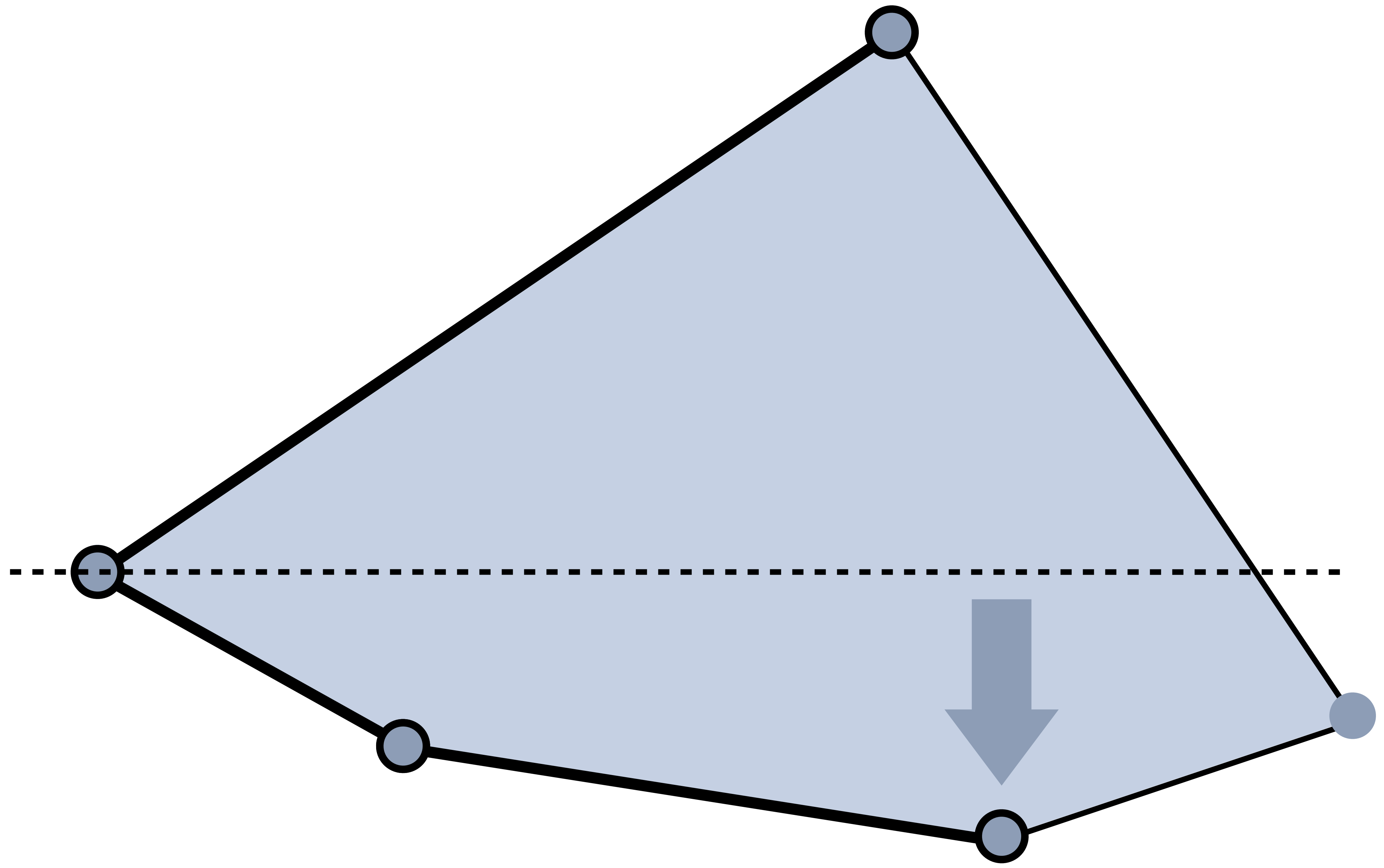


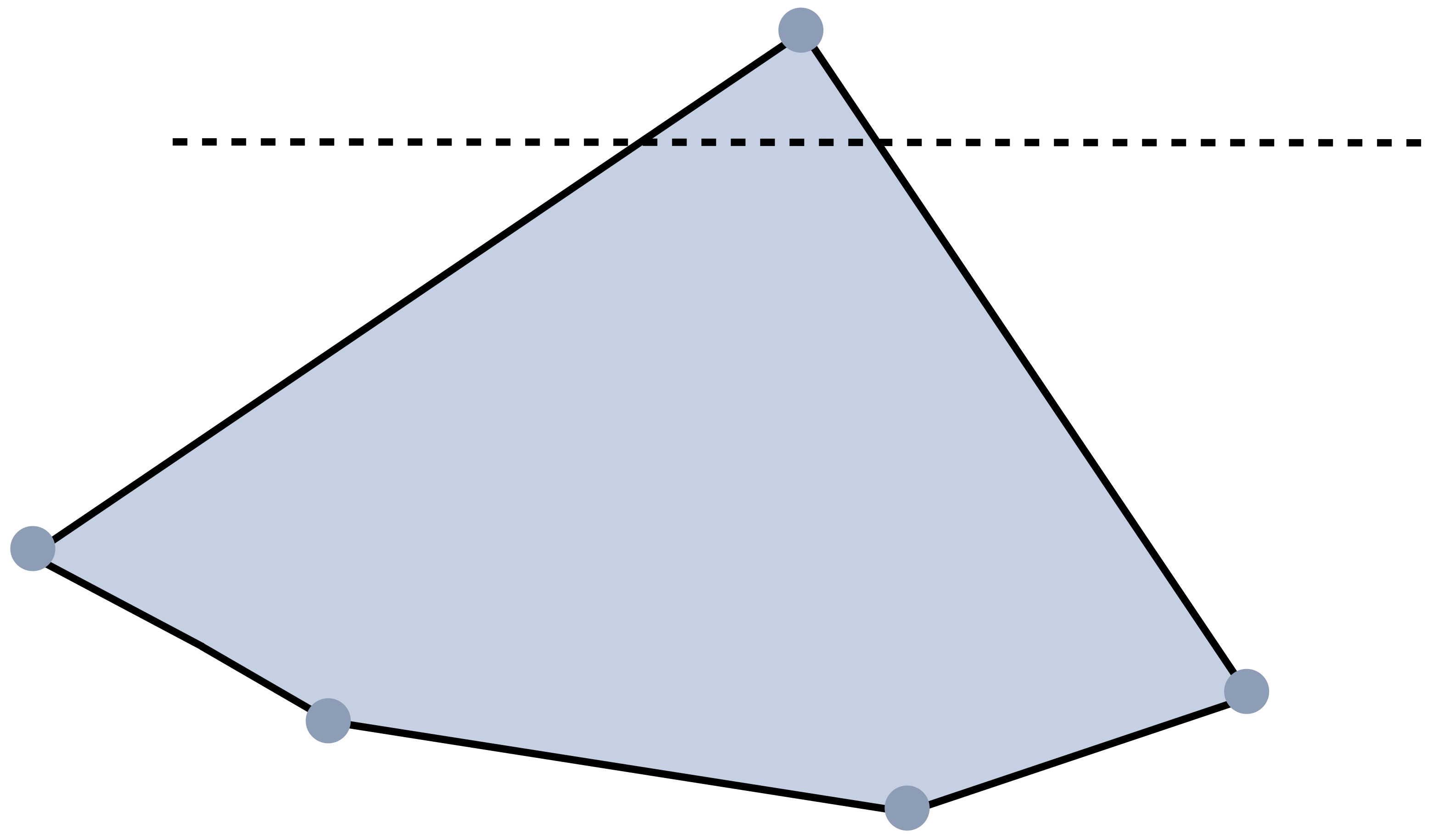


3 Simplex Method: Terminology

Basis. A basis is a subset of n linearly independent constraints and its location is the unique point x satisfying the all n constraints with equality.

Locally Optimal Basis. A basis is locally optimal if its location x is the optimal solution to the LP with the same objective function and *only* the constraints in the basis.

Neighbors. Two bases are neighbors if they have $n - 1$ constraints in common.



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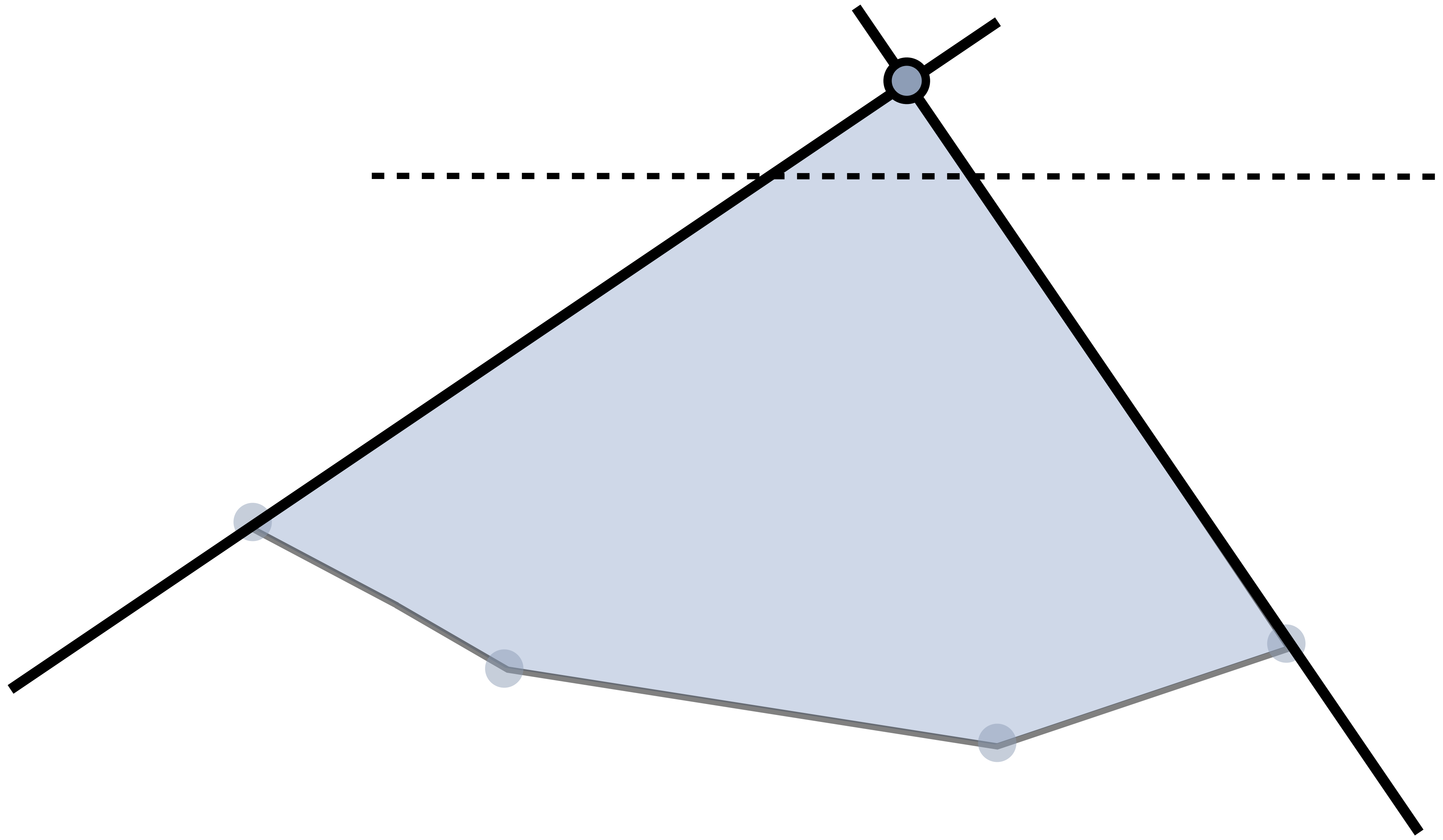

Not locally optimal!

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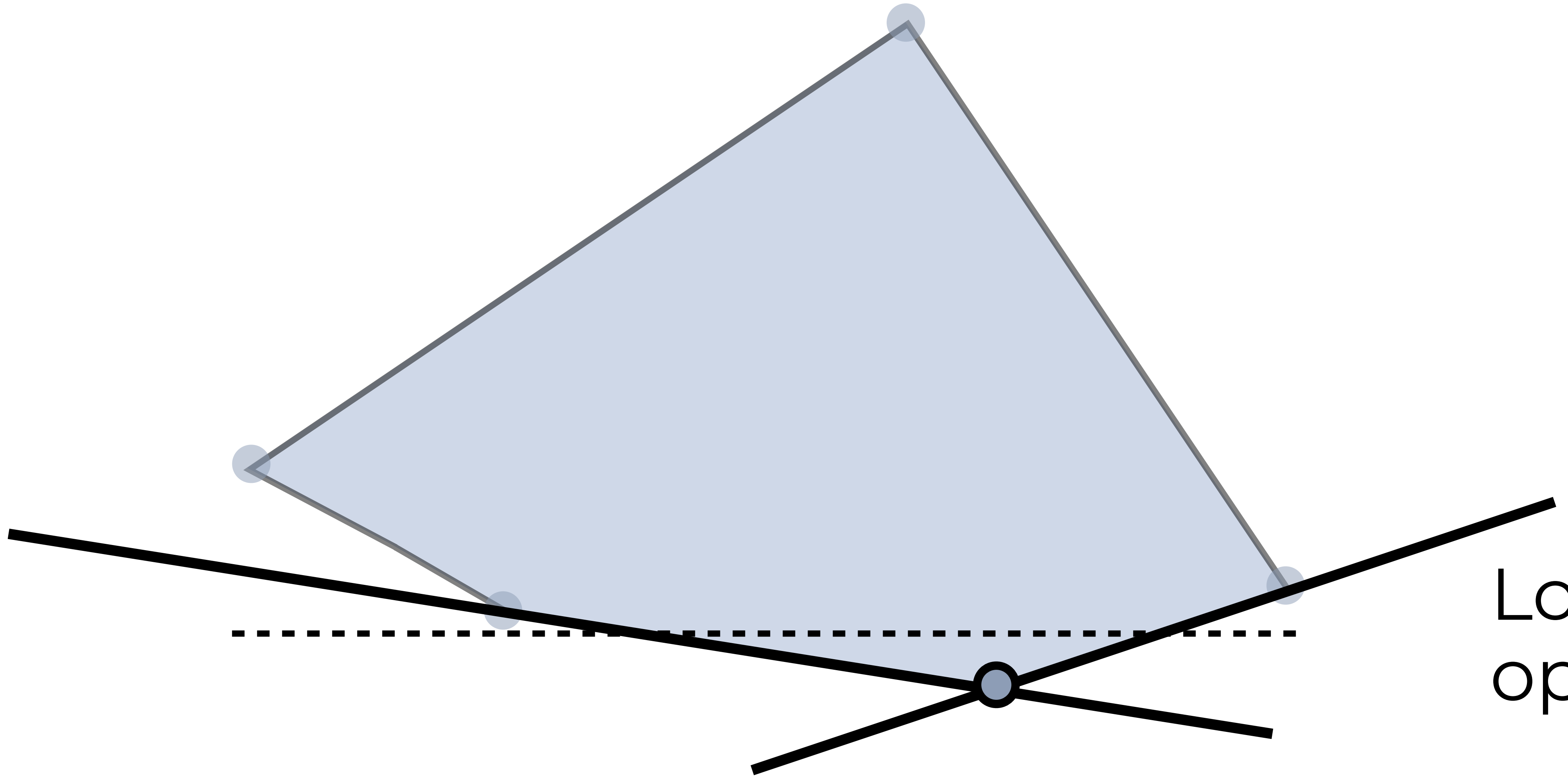

Locally optimal!

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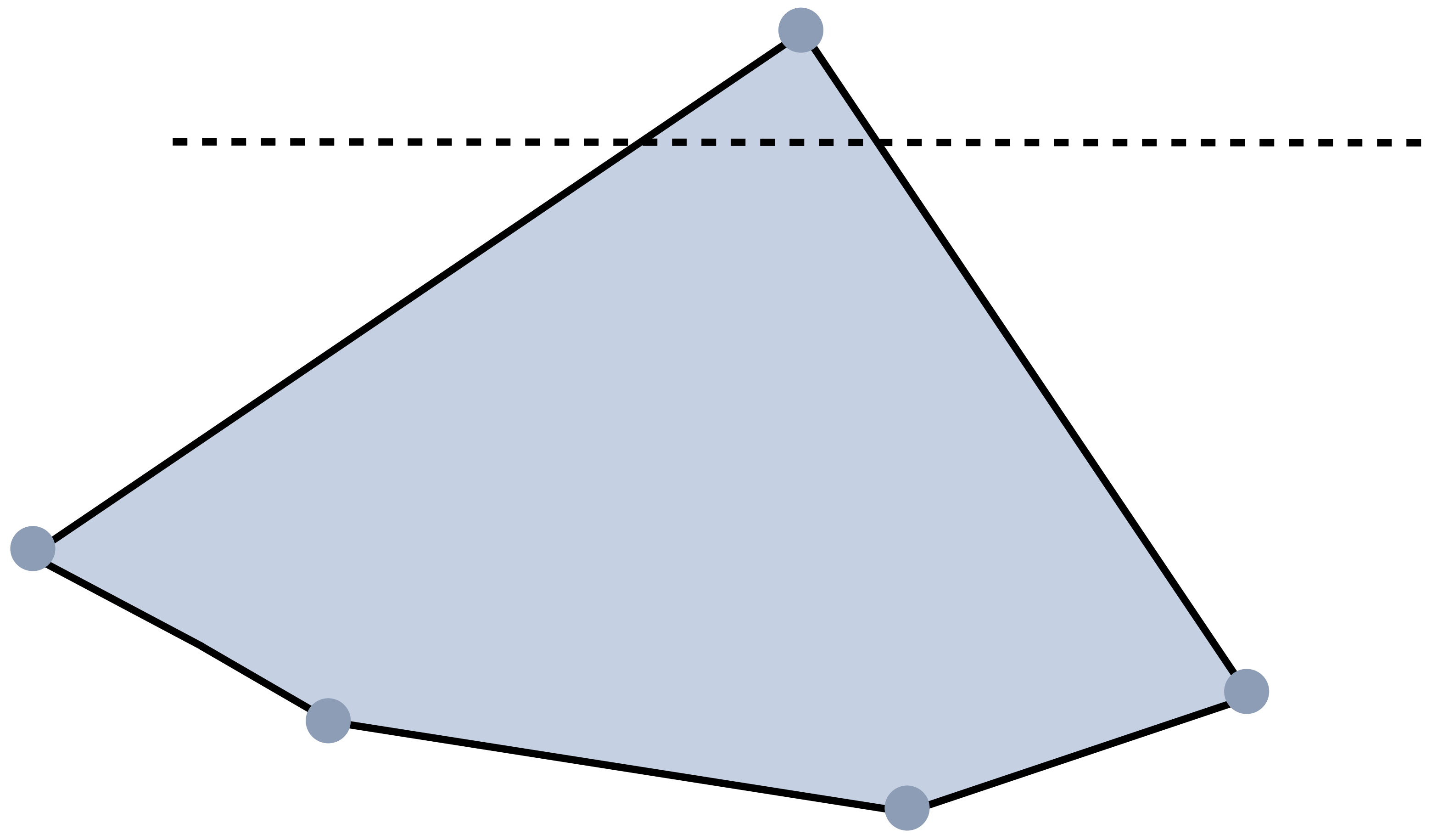

3 Simplex Method

```
Simplex Algorithm($H$).
if $\cap H = \varnothing$
    return INFEASIBLE
$x \leftarrow$ any feasible vertex

while $x$ is not locally optimal:
    $\langle\langle$pivot downward, maintaining feasibility$\rangle\rangle$
    if every feasible neighbor of $x$ is higher than $x$
        return UNBOUNDED
    else
        $x \leftarrow$ any feasible neighbor of that is lower than $x$
return $x$
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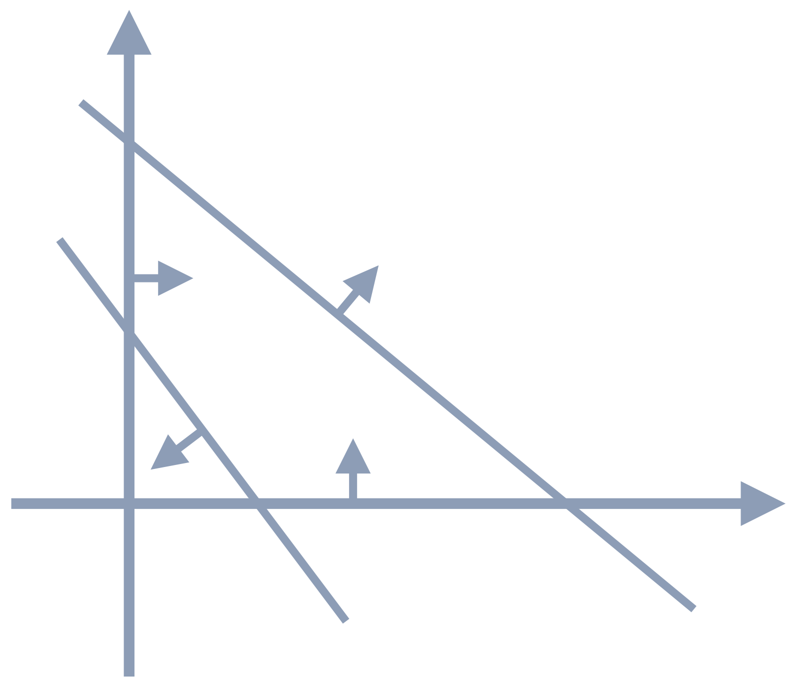

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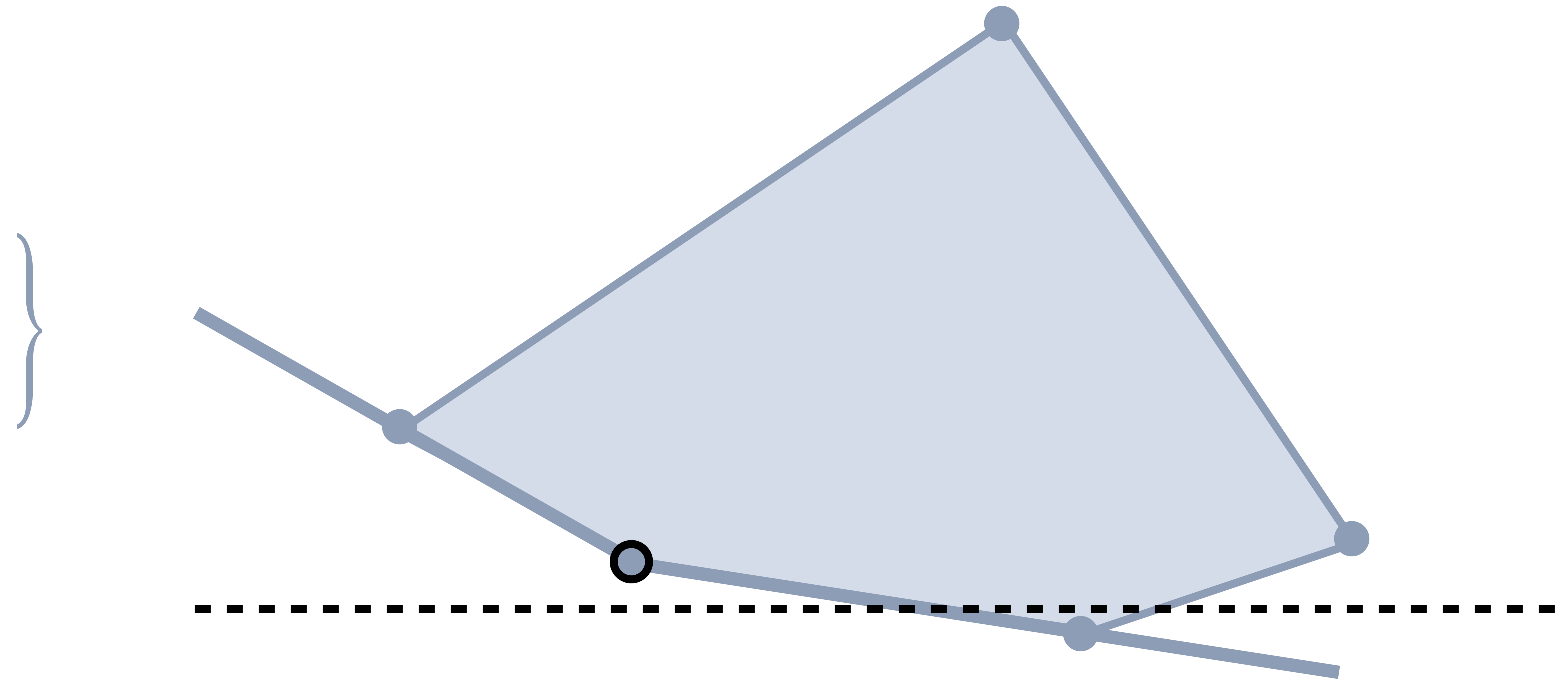

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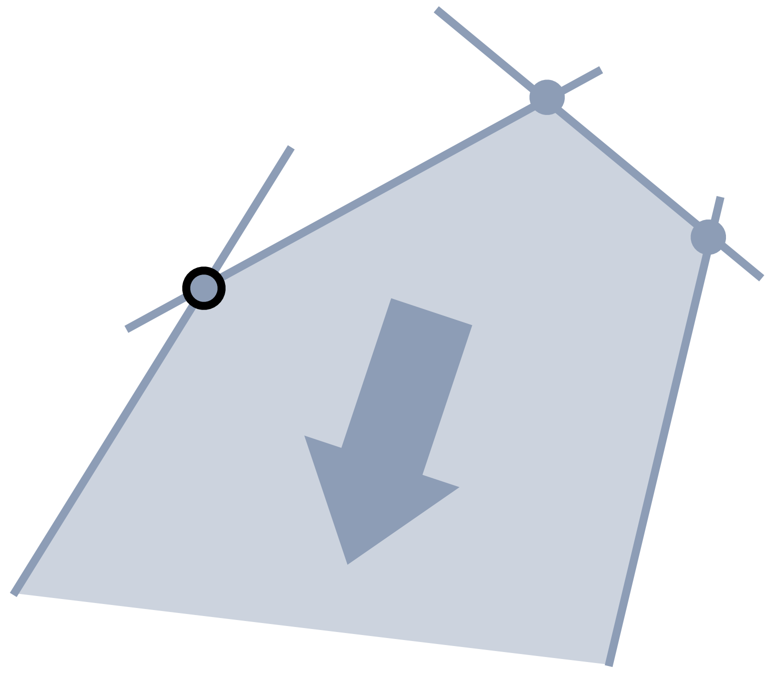

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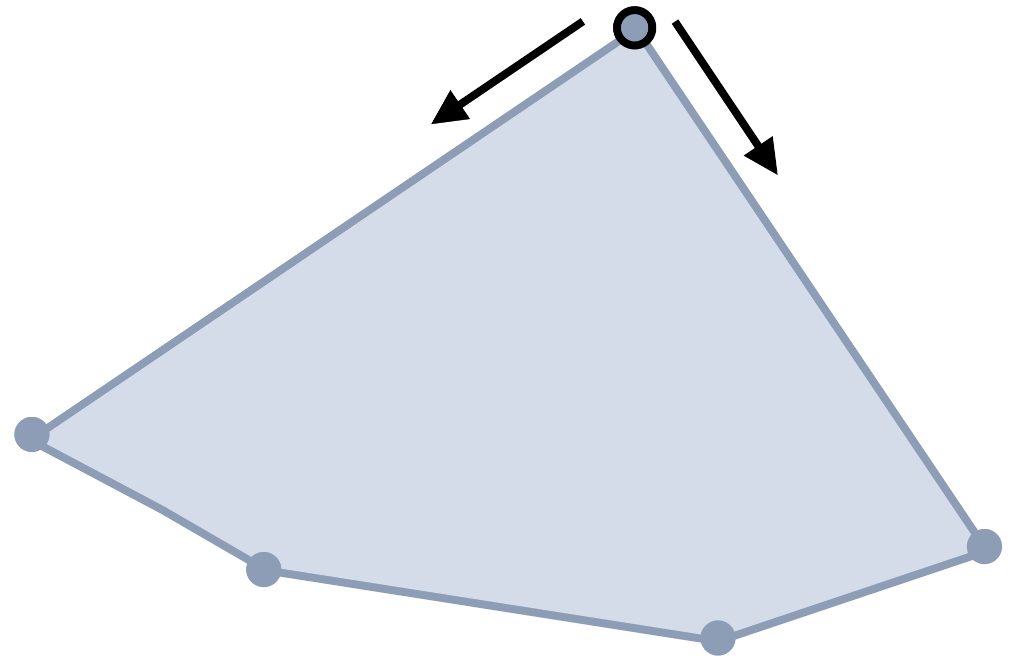
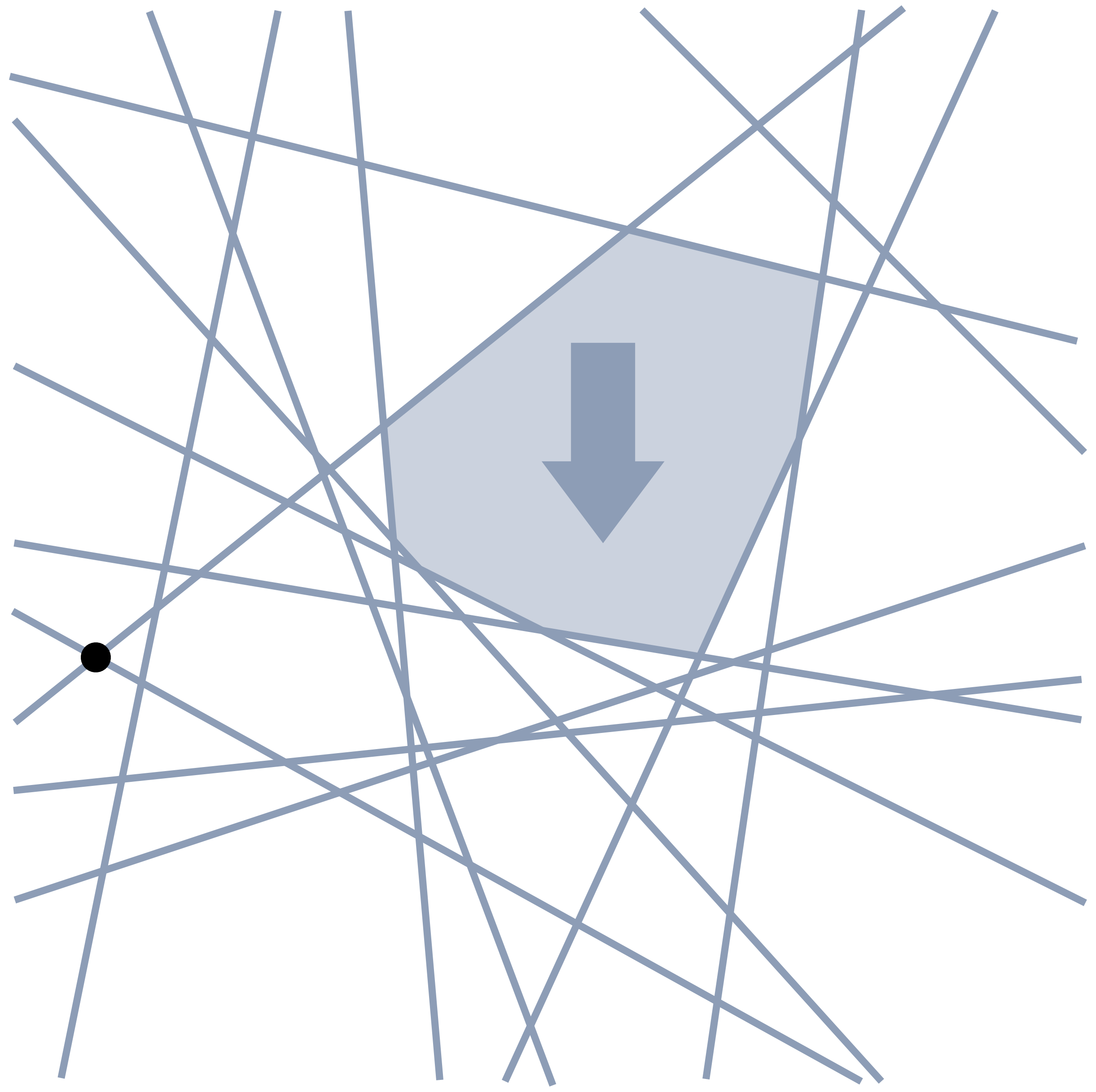


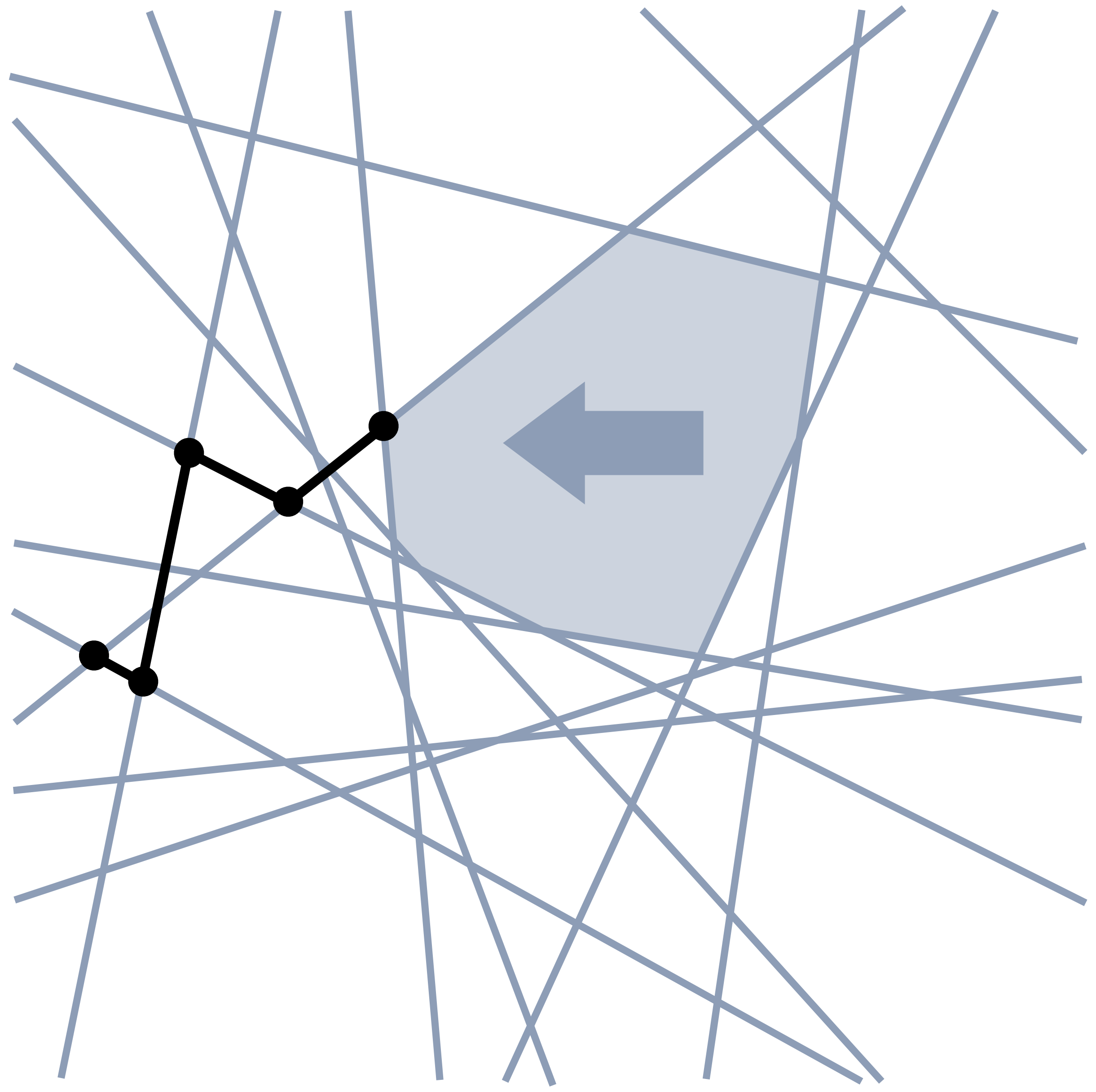
3 Simplex Method: Find Initial Solution

Observations.

1) The feasibility of a vertex does not depend on choice of objective vector.
2) *Every* basis is locally optimal for *some* objective vector.



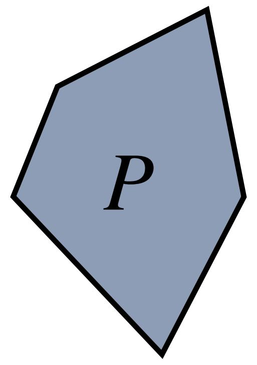
Choose any basis x.



Rotate objective to make x feasible and pivot "up" to a feasible basis.

3 Ellipsoid Algorithm

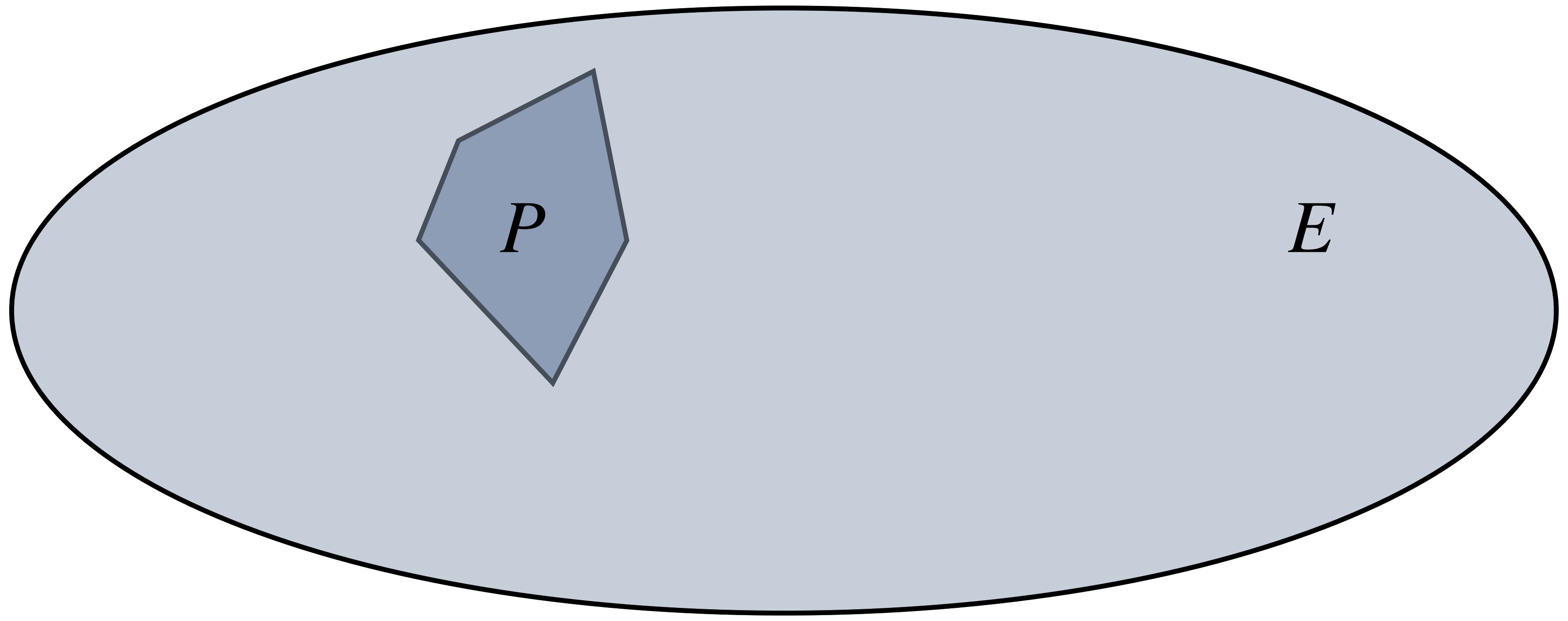
How to find a point in P.



3 Ellipsoid Algorithm

How to find a point in P.

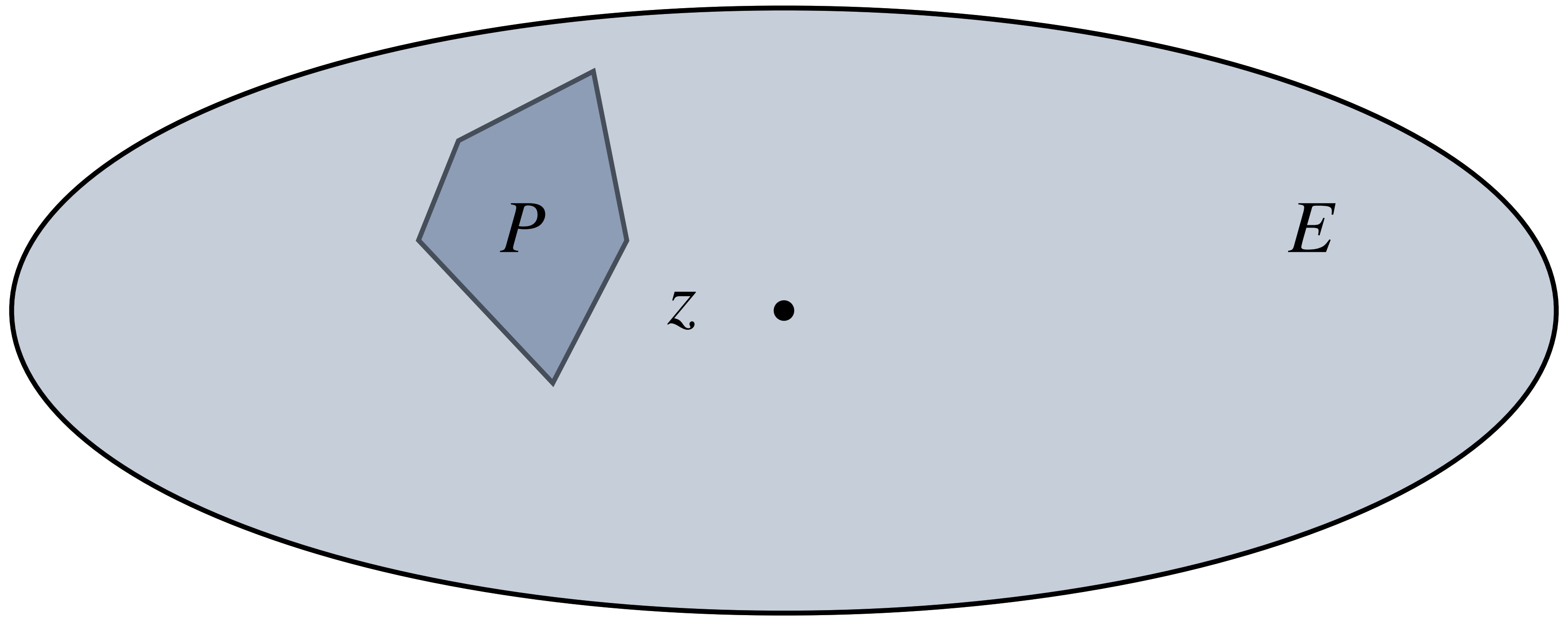
- Maintain an ellipsoid E containing P



Ellipsoid Algorithm

How to find a point in P.

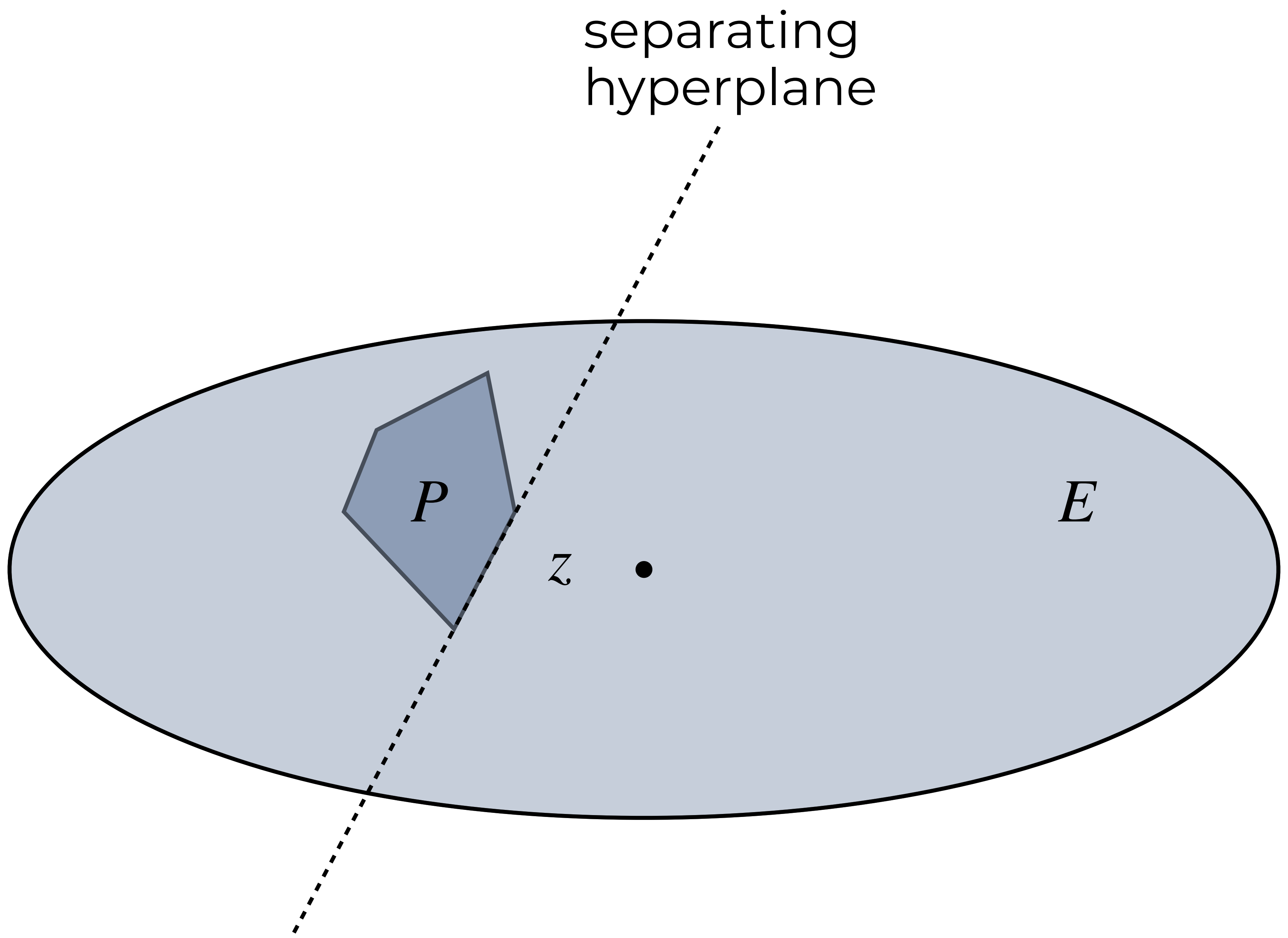
- Maintain an ellipsoid E containing P
- If the center z of ellipsoid is in P, stop;



3 Ellipsoid Algorithm

How to find a point in P.

- Maintain an ellipsoid E containing P
- If the center z of ellipsoid is in P, stop;
  Otherwise find hyperplane separating z from P



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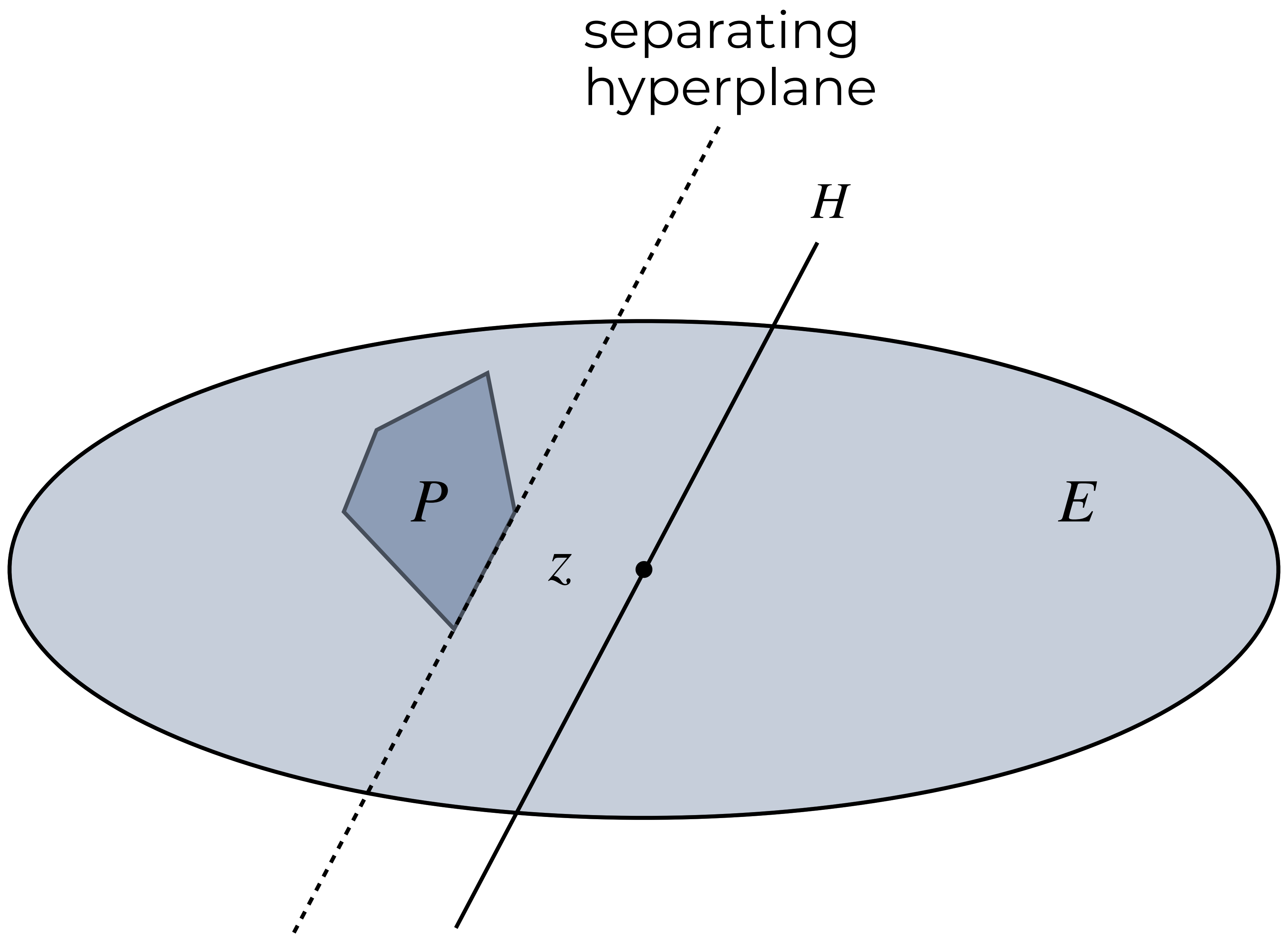
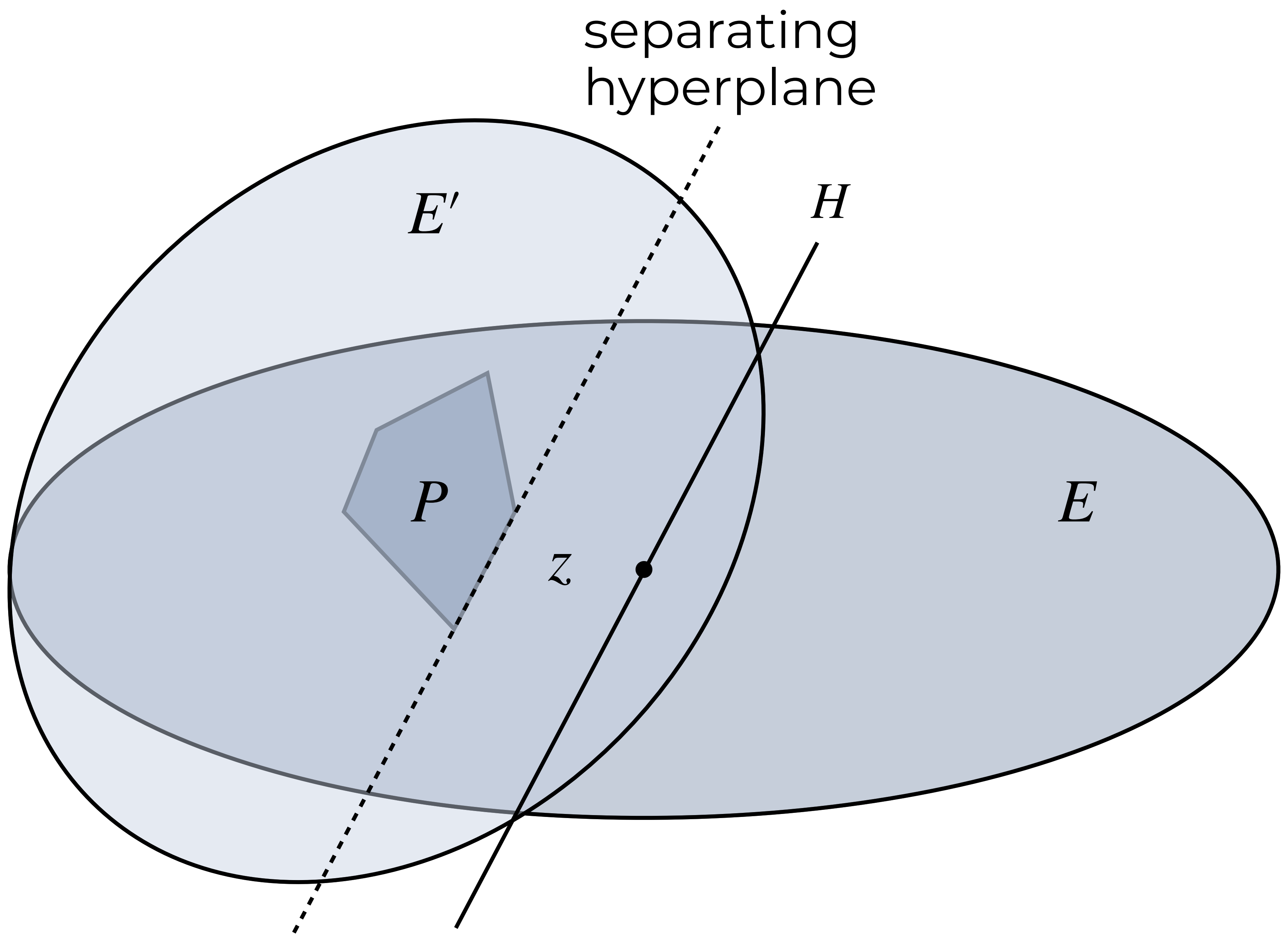


3 Ellipsoid Algorithm

How to find a point in P.

- Maintain an ellipsoid E containing P
- If the center z of ellipsoid is in P, stop;
  Otherwise find hyperplane separating z from P
- Find the smallest ellipsoid E' containing the half-ellipsoid



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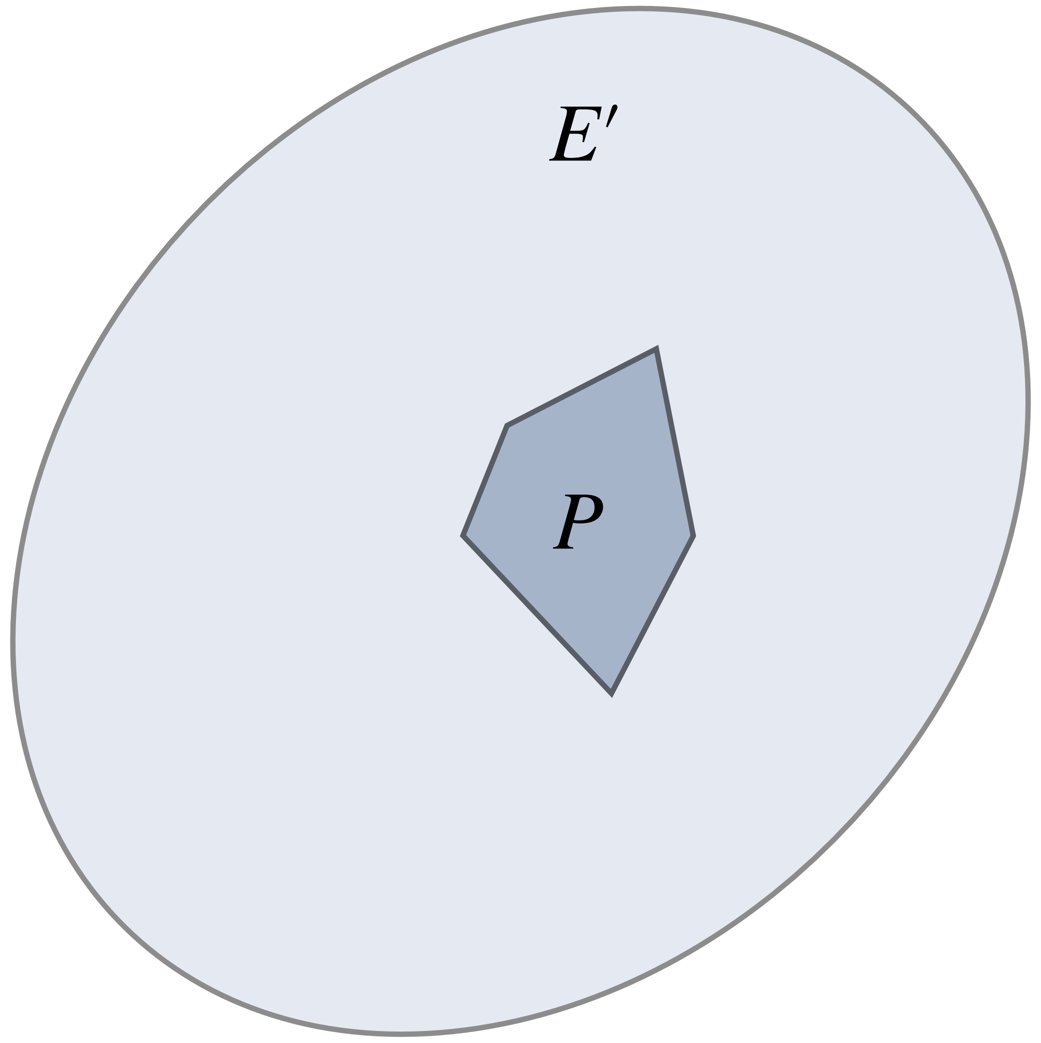
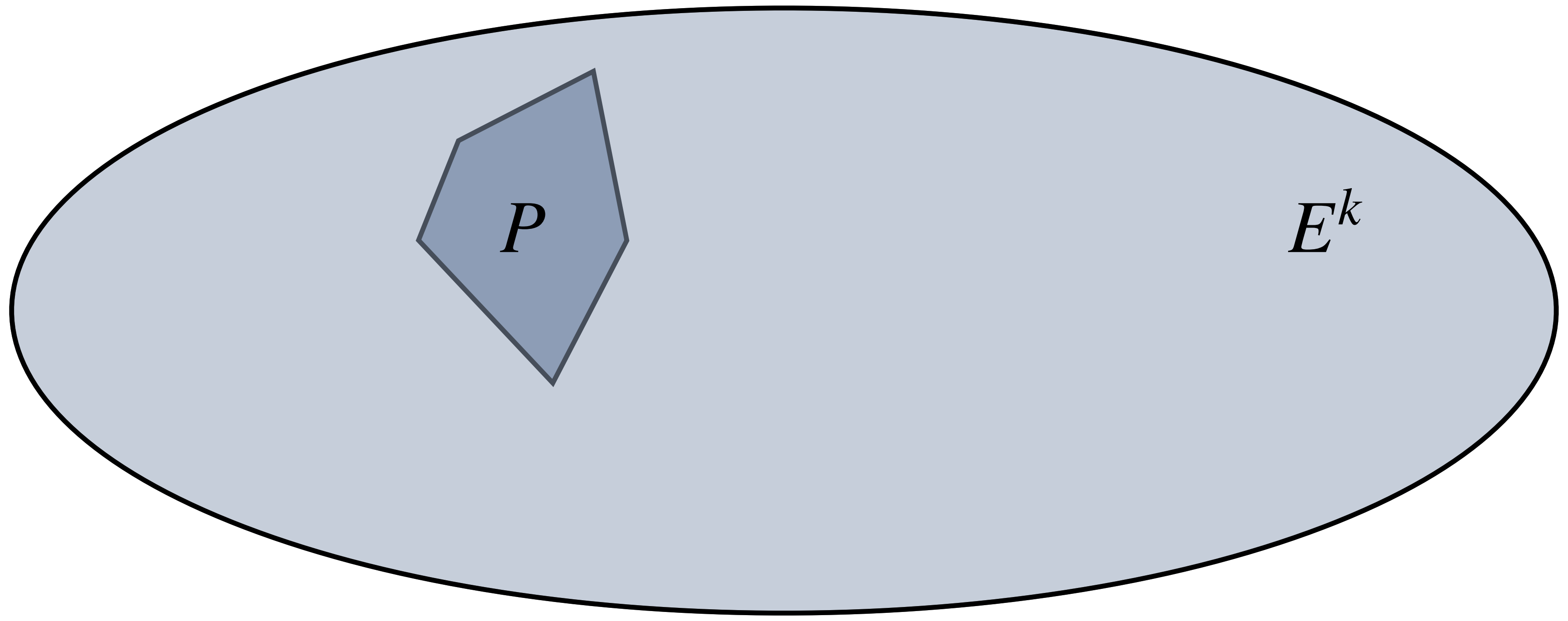
- Repeat with same procedure with E'!



Ellipsoid Algorithm.

Set $k = 0$ and let E^0 be an ellipsoid containing P.



3 Ellipsoid Algorithm

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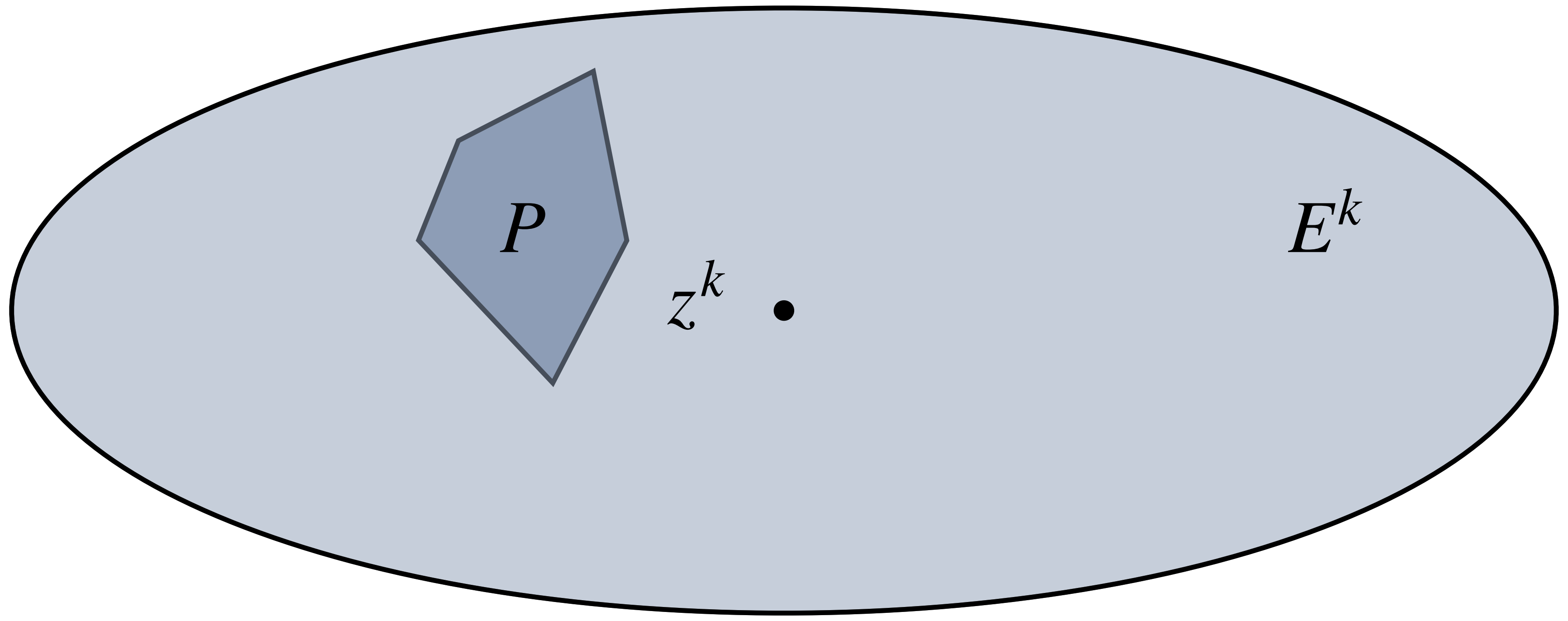
Set $k = 0$ and let E^0 be an ellipsoid containing P.

While center z^k of ellipsoid E^k is not in P:

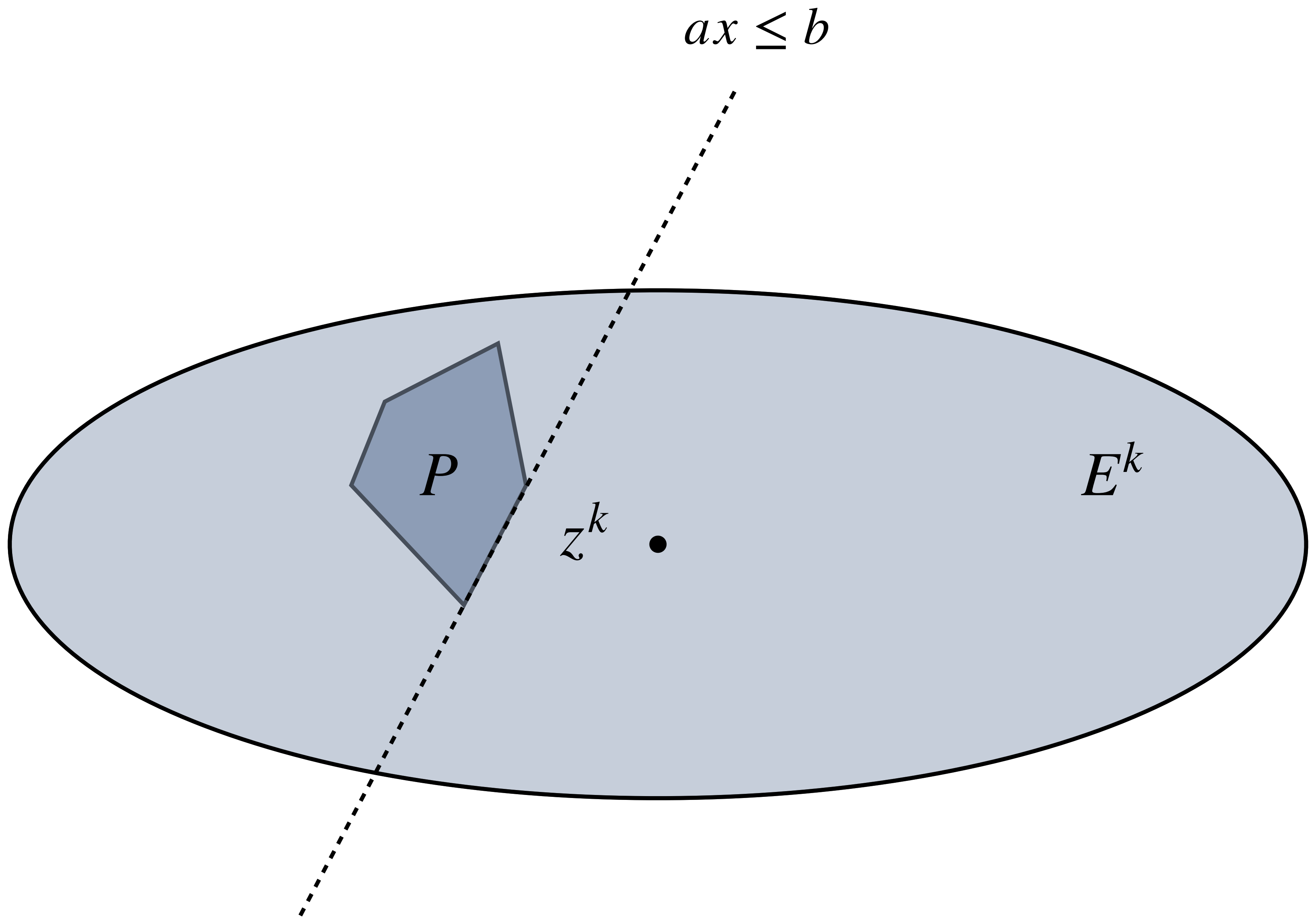


Ellipsoid Algorithm.

Set $k = 0$ and let E^0 be an ellipsoid containing P.

While center z^k of ellipsoid E^k is not in P:

    Find a constraint $ax \leq b$ that is violated by z^k.



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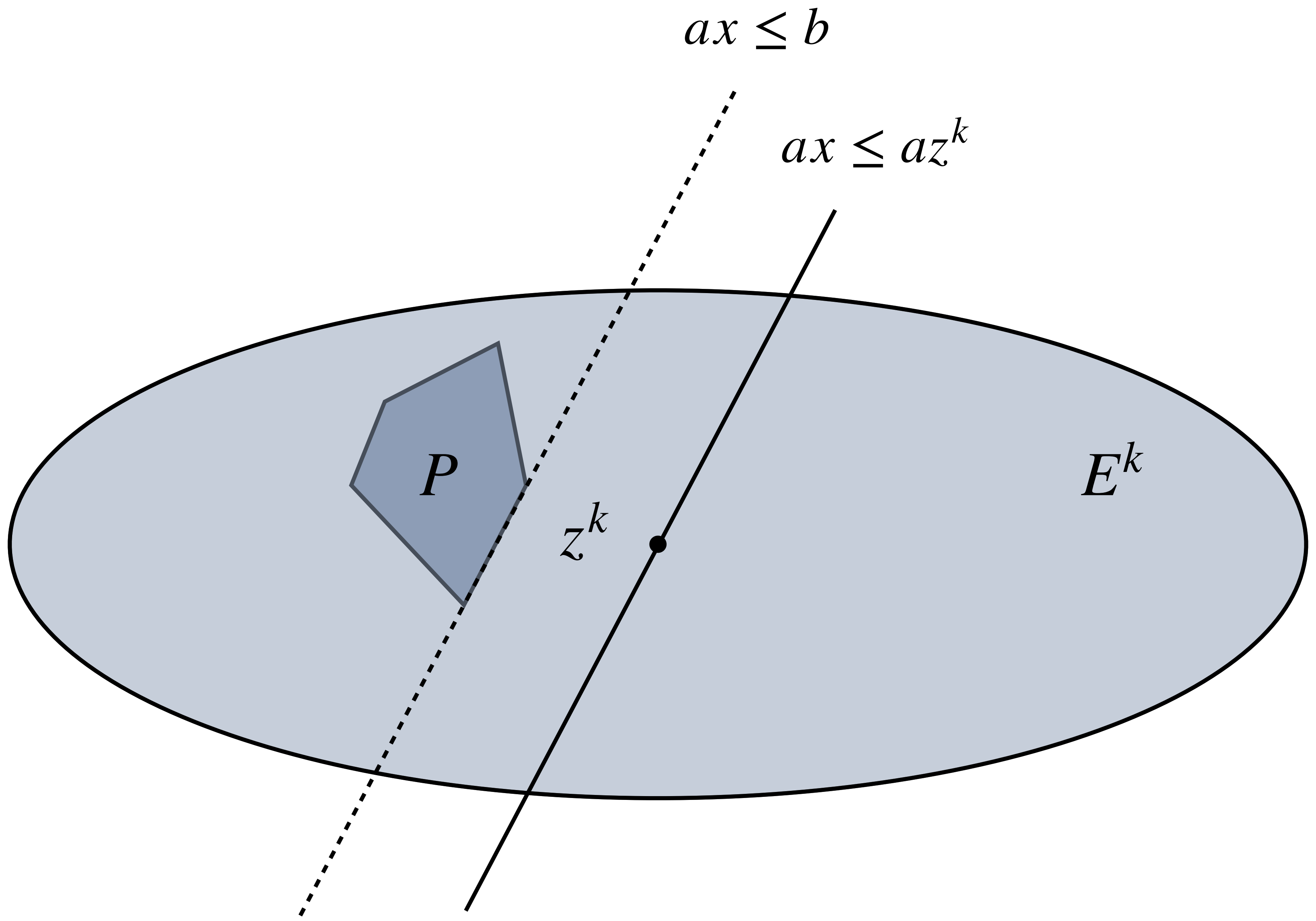

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    Let $E^{k+1}$ be minimum volume ellipsoid containing $E^k \cap \{x : ax \leq az^k\}$.
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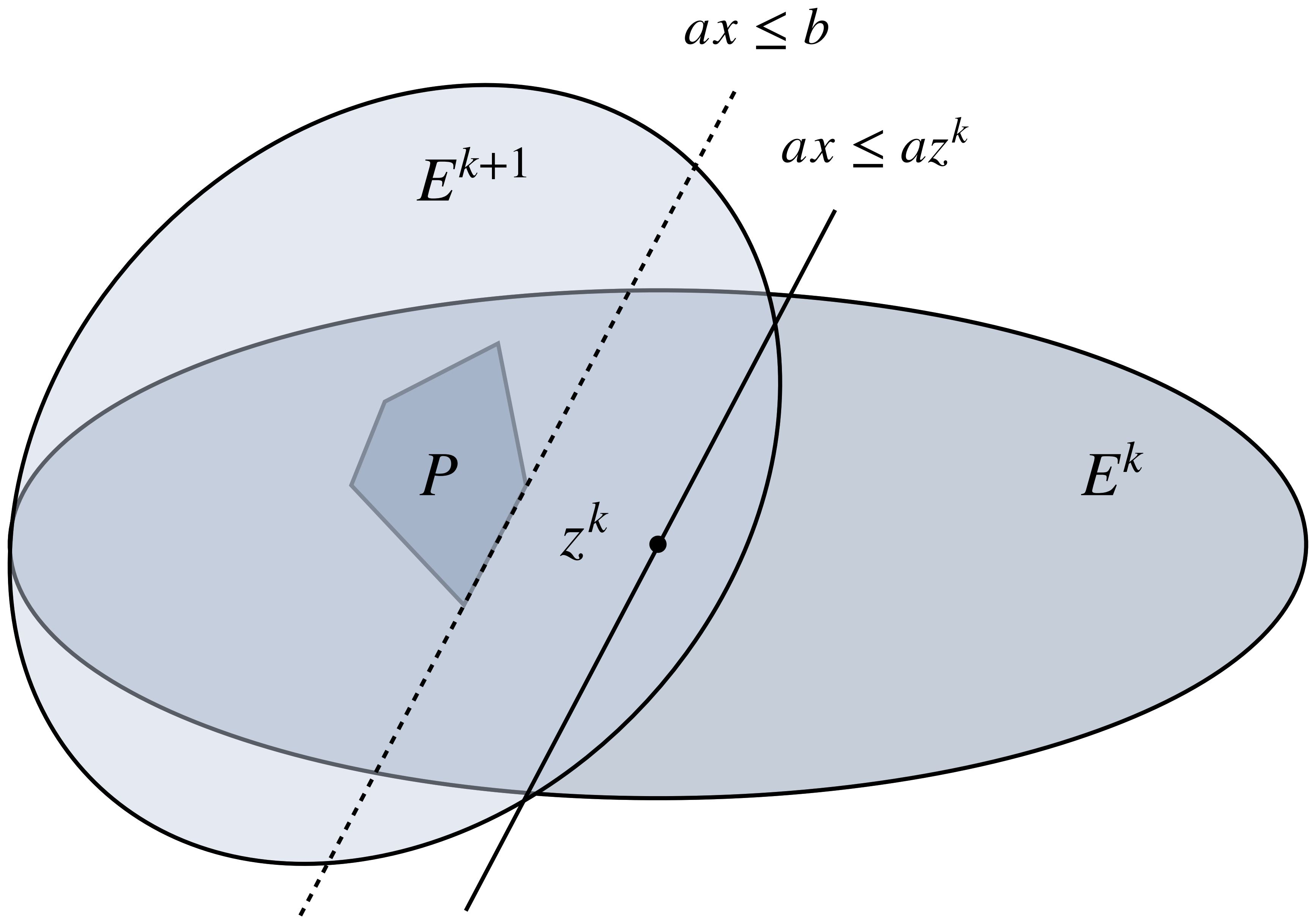

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    Set $k = k + 1$.
```

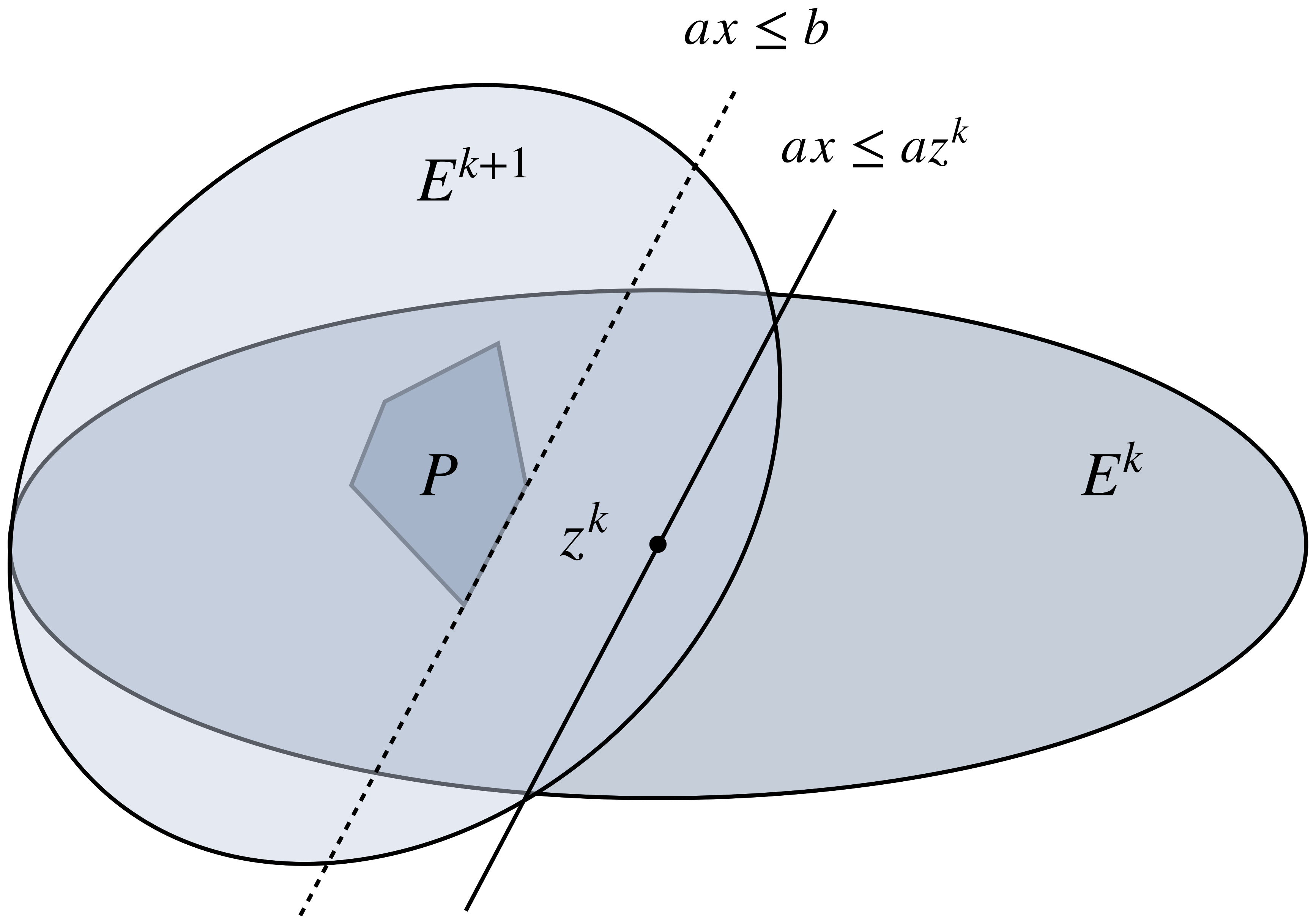
$ax \leq b$
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E^{k+1}
P
z^k
E^k