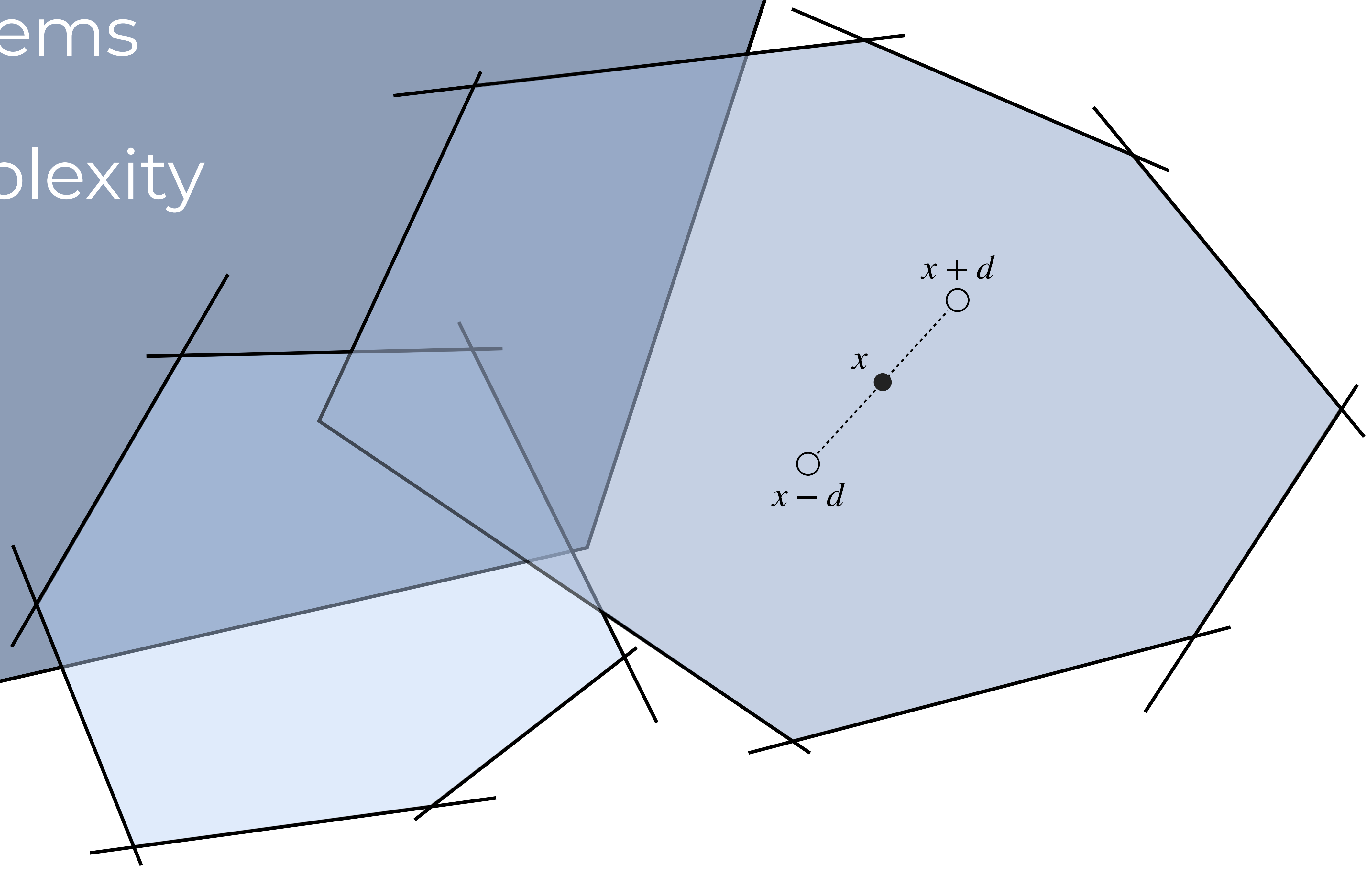
Leo Krull

Linear Programming II

Primal & Dual LP

Duality Theorems

(M)(I)LP Complexity



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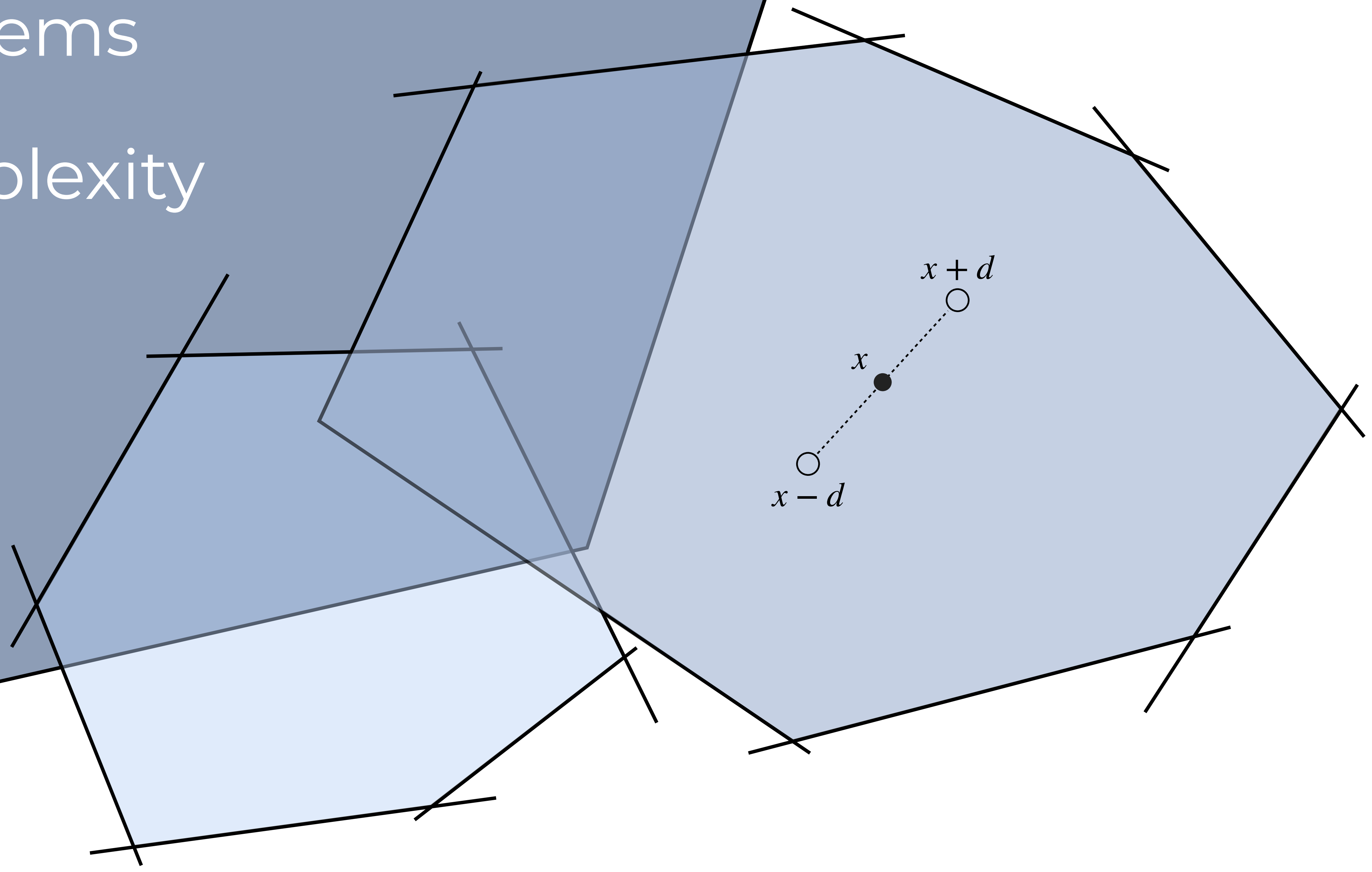

1 Brewery Example: Lower Bound

Question. How to find a lower bound on the optimal value γ^* of the Brewery LP?

$$\begin{aligned} \max \quad & 13x_1 + 23x_2 \\ & 5x_1 + 15x_2 \leq 480 \\ & 4x_1 + 4x_2 \leq 160 \\ & 35x_1 + 20x_2 \leq 1190 \\ & x_1, \quad x_2 \geq 0 \end{aligned}$$

Answer. Any feasible solution to the Brewery LP provides a lower bound.

$$(x_1, x_2) = (34,0) \quad \Rightarrow \quad \gamma^* \geq 442$$
$$(x_1, x_2) = (0,32) \quad \Rightarrow \quad \gamma^* \geq 736$$
$$(x_1, x_2) = (12,28) \quad \Rightarrow \quad \gamma^* \geq 800$$

1 Brewery Example: Upper Bound

Question. Is there a way to prove an upper bound on γ^*?

Answer. Multiply each of the constraints by a new non-negative scalar value y_i .

$$
\begin{array}{lrcl}
\max & 13x_1 + 23x_2 & & \\
 & y_1\,(\;\,5x_1 + 15x_2) & \leq 480 & y_1 \\
 & y_2\,(\;\,4x_1 + \;\,4x_2) & \leq 160 & y_2 \\
 & y_3\,(35x_1 + 20x_2) & \leq 1190 & y_3 \\
 & x_1, \quad x_2 & \geq 0 &
\end{array}
$$

Any feasible solution (x_1, x_2) must satisfy all the inequalities, so it must also satisfy their sum.

$$y_1(5x_1 + 15x_2) + y_2(4x_1 + 4x_2) + y_3(35x_1 + 20x_2) \leq 480y_1 + 160y_2 + 1190y_3$$

1 Brewery Example: Upper Bound

Suppose that the coefficient of each variable x_i is larger than the corresponding coefficient of the objective function.

$$x_1(5y_1 + 4y_2 + 35y_3) \quad + \quad x_2(15y_1 + 4y_2 + 20y_3) \leq 480y_1 + 160y_2 + 1190y_3$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$(5y_1 + 4y_2 + 35y_3) \geq 13, \qquad (15y_1 + 4y_2 + 20y_3) \geq 23 \tag{1}$$

This assumption implies an upper bound on the objective value of *any* feasible solution.

$$\begin{aligned} 13x_1 + 23x_2 &\leq x_1(5y_1 + 4y_2 + 35y_3) + x_2(15y_1 + 4y_2 + 20y_3) \\ &\leq 480y_1 + 160y_2 + 1190y_3 \end{aligned} \tag{2}$$

1 Brewery Example: Upper Bound

In particular, by plugging in the optimal solution (x_1^*, x_2^*) for the original LP, the following upper bound on γ^* can be obtained.

$$\gamma^* = 13x_1^* + 23x_2^* \leq 480y_1 + 160y_2 + 1190y_3$$

Question. How tight can this upper bound be? That is, how small can the expression $480y_1 + 160y_2 + 1190y_3$ be without violating any of the inequalities (1) used to prove the upper bound?

Answer. This can be expressed as another linear program!

1 Brewery Example: Upper Bound

Question. What does this linear program look like?

Answer. It is a minimization problem that combines the expressions (1) and (2) with non-negativity constraints for y_1, y_2 and y_3.

$$\begin{aligned}
\min\ \ 480y_1 + 160y_2 + 1190y_3 & \\
5y_1 + 4y_2 + 35y_3 &\geq 13 \\
15y_1 + 4y_2 + 20y_3 &\geq 23 \\
y_1,\ y_2,\ y_3 &\geq 0
\end{aligned}$$

Observation. While the original Brewery LP has 2 variables and 3 constraints, the above LP has 3 variables and 2 constraints.

1 Brewery Example: Economic Interpretation

Brewer. Find optimal mix of beer and ale to maximize profits.

$$\begin{aligned} \max \quad & 13x_1 + 23x_2 \\ & 5x_1 + 15x_2 \leq 480 \\ & 4x_1 + 4x_2 \leq 160 \\ & 35x_1 + 20x_2 \leq 1190 \\ & x_1, \quad x_2 \geq 0 \end{aligned}$$

Primal Problem

Entrepreneur. Buy individual resources from the brewer to minimize costs.

$$\begin{aligned} \min \quad & 480y_1 + 160y_2 + 1190y_3 \\ & 5y_1 + 4y_2 + 35y_3 \geq 13 \\ & 15y_1 + 4y_2 + 20y_3 \geq 23 \\ & y_1, \quad y_2, \quad y_3 \geq 0 \end{aligned}$$

Dual Problem

1 Primal and Dual LP

Dual Problem. Every linear program, referred to as the primal problem, has a corresponding dual problem, which provides an upper bound to the optimal value of the primal problem.

$$\begin{aligned} \max\ & c^T x \\ & Ax \le b \qquad (P) \\ & x \ge 0 \end{aligned}$$

Primal Problem

$$\begin{aligned} \min\ & y^T b \\ (D) \qquad & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

Dual Problem

1 Primal and Dual LP

Lemma. The dual of the dual of any linear program is always (equivalent to) the original linear program.

$$
\begin{array}{lcl}
\min & y^T b \\
 & A^T y \geq c \\
 & y \geq 0
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{lcl}
\max & -y^T b \\
 & -A^T y \leq -c \\
 & y \geq 0
\end{array}
\quad \overset{}{\underset{\text{dualize}}{\Longrightarrow}} \quad
\begin{array}{lcl}
\min & -c^T x \\
 & -(A^T)^T x \geq -b \\
 & x \geq 0
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{lcl}
\max & c^T x \\
 & A x \leq b \\
 & x \geq 0
\end{array}
$$

(D) max form of (D) dual of (D) (P)

Rewrite the dual as a maximization problem in canonical form and take the dual.

1 Construct LP Dual

Construction. Given a primal (P) that is not in canonical form, the dual (D) can be derived by converting (P) into canonical form and applying the rules below.

Primal (P)	maximize	minimize	Dual (D)
Constraints	$a_i x = b_i$	y_i unrestricted	Variables
	$a_i x \leq b_i$	$y_i \geq 0$	
	$a_i x \geq b_i$	$y_i \leq 0$	
Variables	$x_j \geq 0$	$\alpha_j^T y \geq c_j$	Constraints
	$x_j \leq 0$	$\alpha_j^T y \leq c_j$	
	x_j unrestricted	$\alpha_j^T y = c_j$	

Notation: a_i refers to the i-th row of A and α_j to the j-th column of A.

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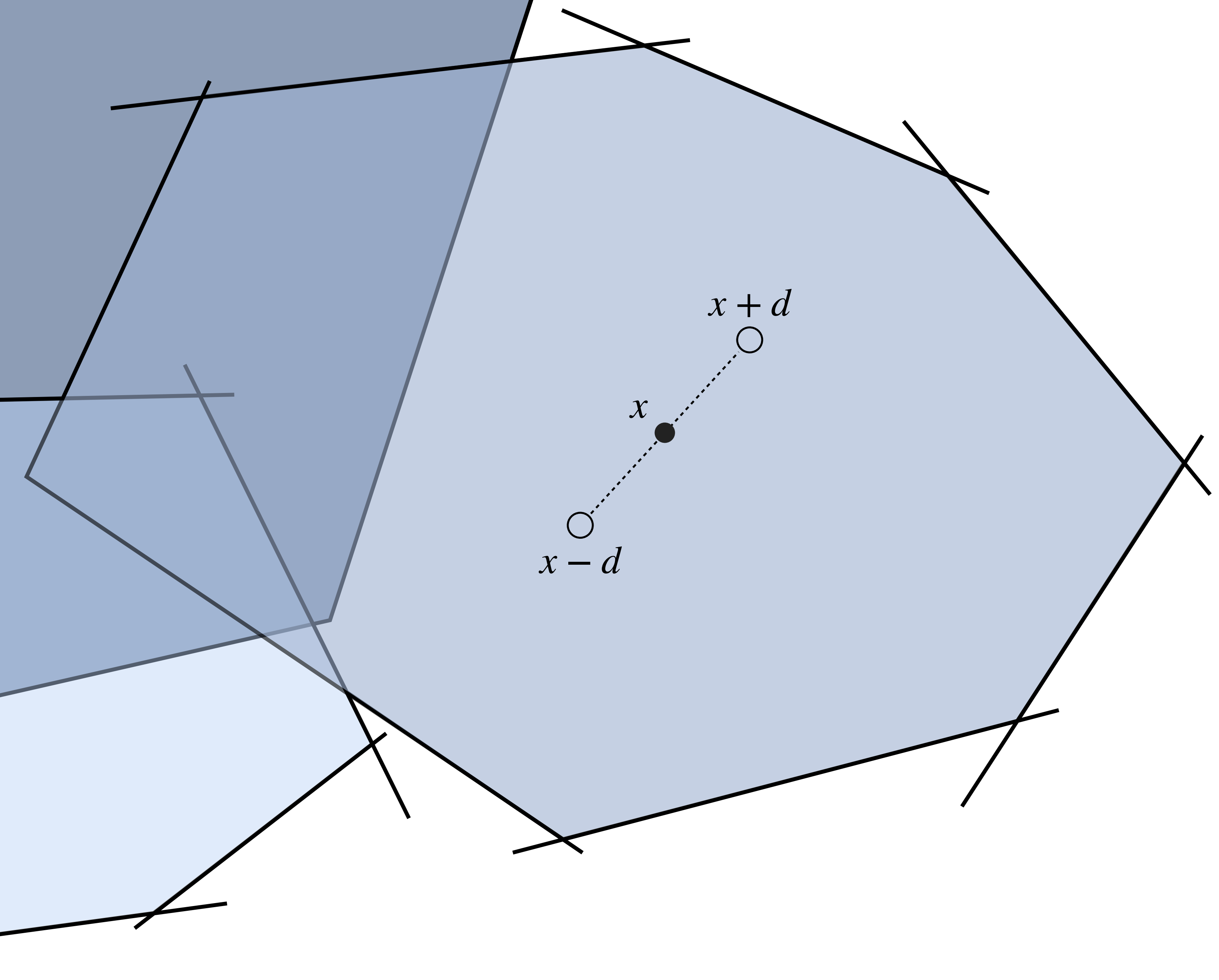


2 Solvability of Systems of Inequalities

Farkas Lemma. The system $Ax = b, x \geq 0$ has no solution, if and only if there exists y with $A^T y \geq 0$ and $b^T y < 0$.

Simply Put. Given a matrix A and a right-hand side b, one of the following two systems is feasible while the other one is infeasible.

$$\text{(1)} \quad \begin{aligned} &\exists x \in \mathbb{R}^n \\ &\text{s.t. } Ax = b, \\ &\qquad x \geq 0 \end{aligned}$$

$$\text{(2)} \quad \begin{aligned} &\exists y \in \mathbb{R}^m \\ &\text{s.t. } A^T y \geq 0, \\ &\qquad b^T y < 0 \end{aligned}$$

2 Farkas Lemma: Example

Example. Consider the solutions to two different systems for

$$A = \begin{pmatrix} 4 & 4 \\ 3 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

(1)

$$\begin{aligned} 4x_1 + 4x_2 &= 5 \\ 3x_1 \quad\quad &= 3 \\ x_1, \quad x_2 &\geq 0 \end{aligned} \quad\Rightarrow\quad \begin{aligned} x_1 &= 1 \\ x_2 &= \frac{1}{4} \end{aligned}$$

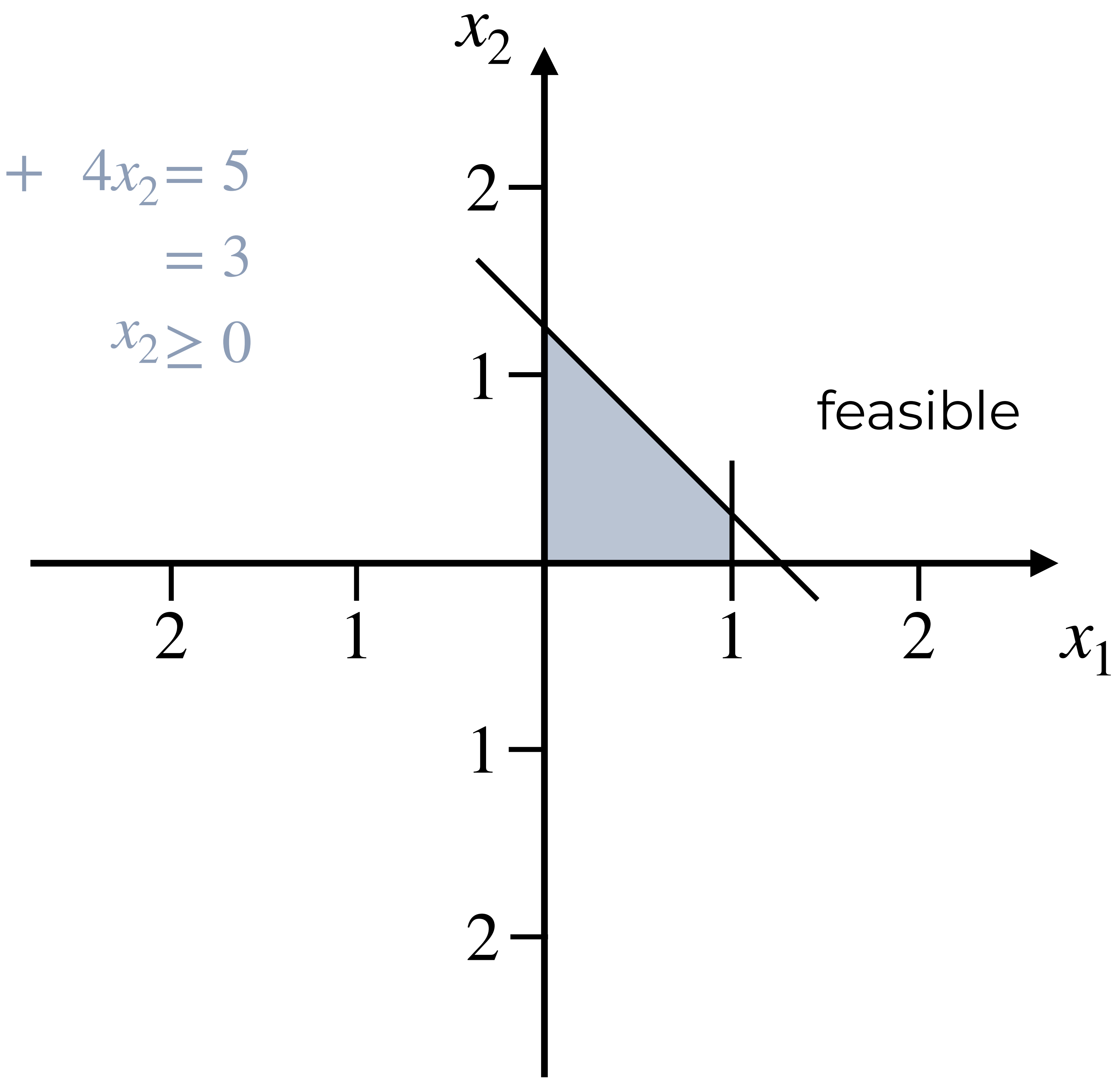
(2)

$$\begin{aligned} 4y_1 + 3y_2 &\geq 0 \\ 4y_1 \quad\quad &\geq 0 \\ 5y_1 + 3y_2 &< 0 \end{aligned} \quad\Rightarrow\quad \begin{aligned} 4y_1 + 3y_2 &\geq 0 \\ y_1 \quad\quad &\geq 0 \\ 5y_1 + \underbrace{3y_2}_{> -5y_1} &< 0 \end{aligned} \quad ↯$$

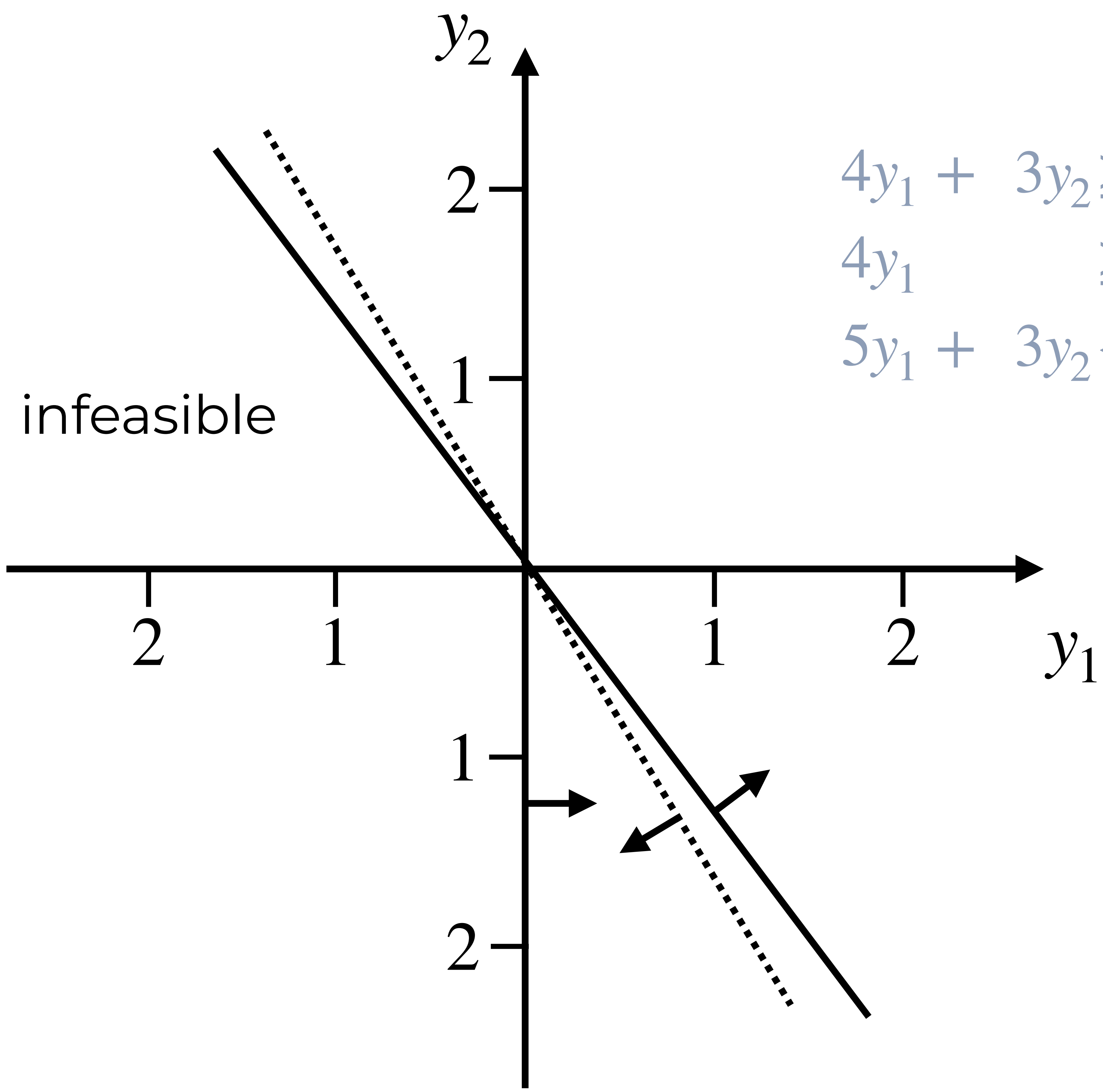
2 Farkas Lemma: Example

Example. Drawing the constraints shows the feasibility and infeasibility of the two systems.

$$\begin{aligned} 4x_1 + 4x_2 &= 5 \\ 3x_1 &= 3 \\ x_1, \quad x_2 &\geq 0 \end{aligned}$$



$$\begin{aligned} 4y_1 + 3y_2 &\geq 0 \\ 4y_1 &\geq 0 \\ 5y_1 + 3y_2 &< 0 \end{aligned}$$



2 Solvability of Systems of Inequalities

Farkas Lemma. The system $Ax = b, x \geq 0$ has no solution, if and only if there exists y with $A^T y \geq 0$ and $b^T y < 0$.

Theorem of the Alternatives. The system $Ax \leq b$ has no solution $x \in \mathbb{R}^n$, if and only if there exists $y \in \mathbb{R}^m$ such that $y \geq 0$, $A^T y = 0$ and $b^T y < 0$.

Proof. [partly] Both systems cannot have solution, since otherwise holds that

$$0 > b^T y = y^T b \geq y^T A x = 0^T x = 0.$$

2 Weak Duality

Weak Duality. If x is a feasible solution to (P) and y is a feasible solution to its dual (D), then it holds that $c^T x \leq b^T y$.

Proof. Since both x and y are feasible, it holds that $Ax \leq b, x \geq 0$ and $A^T y \geq c$, $y \geq 0$. Hence it follows

$$c^T x \leq (A^T y)^T x = y^T A x \leq b^T y \, . \quad \square$$

2 Weak Duality

Implications. The Weak Duality Theorem has three important consequences:

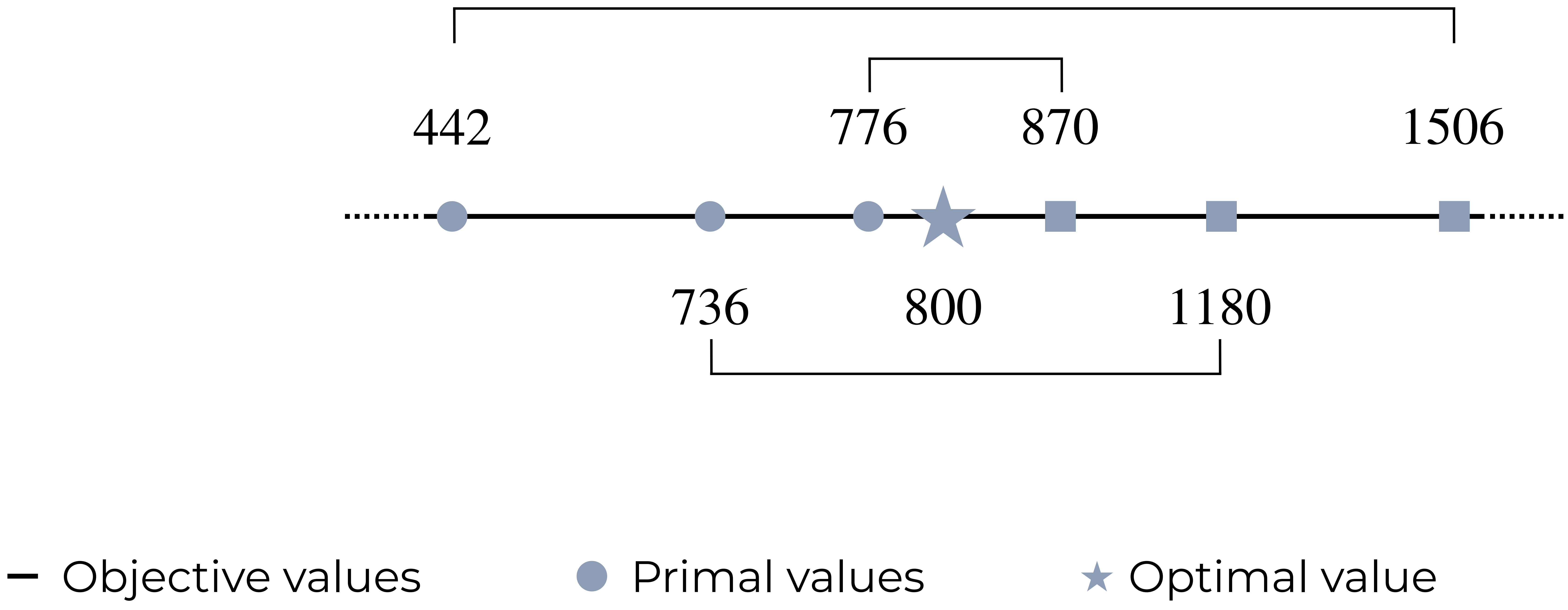
- If $c^T x = b^T y$, then x and y are *optimal* primal and dual solutions, respectively.
- If a linear program is unbounded, then its dual is infeasible.
- If a linear program is feasible, then its dual is bounded.

		Dual (D)		
		Finite optimum	Unbounded	Infeasible
Primal (P)	Finite optimum	✓	✗	✗
	Unbounded	✗	✗	✓
	Infeasible	✗	✓	✓

✓ Possible ✗ Impossible

2 Duality Gap

Duality Gap. Let x be a feasible solution to the primal (P) and y be a feasible solution to the dual (D), then the duality gap is equal to $c^T x - b^T y$ and describes the difference between the primal and dual solutions.



2 Strong Duality

Strong Duality. If $x*$ is an optimal solution to (P), then there exists an optimal solution $y*$ for its dual (D) such that $c^T x* = b^T y*$.

Proof Game Plan.

- Write a big system of inequalities in x and y such that
    - (i) x is primal feasible
    - (ii) y is dual feasible
    - (iii) $c^T x \geq b^T y$
- Use the **Theorem of the Alternatives** or **Farkas Lemma** to show that the infeasibility of this system of inequalities would contradict the feasibility of either (P) or (D)

Strong Duality

Strong Duality. If x^* is an optimal solution to (P), then there exists an optimal solution y^* for its dual (D) such that $c^T x^* = b^T y^*$.

Proof. Let x' be a feasible solution of (P) and y' a feasible solution of (D).
By contradiction, suppose that there are no solutions $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ with $c^T x \geq b^T y$, hence the following system is infeasible.

$$
\begin{array}{rrcll}
\text{primal} & Ax & \leq & b & \} \; s \\
\text{dual} & -A^T y & \leq & -c & \} \; t \\
y \geq 0 & -Iy & \leq & 0 & \} \; u \\
c^T x \geq b^T y & -c^T x + b^T y & \leq & 0 & \} \; v
\end{array}
$$

2 Strong Duality

Proof. Using the **Theorem of the Alternatives**, there must exist $s \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}$ with $s, u, v \geq 0$ and $z^T = (s, t, u, v)$ such that

$$\underbrace{\begin{pmatrix} A & 0 \\ 0 & -A^T \\ 0 & -I \\ -c^T & b^T \end{pmatrix}}_{\mathscr{A}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \leq \underbrace{\begin{pmatrix} b \\ -c \\ 0 \\ 0 \end{pmatrix}}_{b'} \implies \underbrace{\begin{pmatrix} A & 0 & 0 & -c \\ 0 & -A & -I & b \end{pmatrix}}_{\mathscr{A}^T} \cdot \underbrace{\begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}}_{z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2 Strong Duality

Proof. Combining the system $\mathscr{A}^T z = 0$ with $(b')^T z < 0$ yields the following system of (in-)equalities

$$\begin{aligned} A^T s \; - \qquad\qquad\qquad\qquad cv &= 0 \\ -At \; - u + bv &= 0 \\ b^T s \; - \; c^T t \qquad\qquad\qquad &< 0 \end{aligned}$$

In order to show that this system contradicts the feasibility of either (P) or (D), there are two different cases depending on the value of v.

Proof. Case 1: $v > 0$

By dividing the equations by v and renaming all the variables, there exist $s', u' \geq 0$ with $s' = \frac{1}{v}s, \; t' = \frac{1}{v}t, \; u' = \frac{1}{v}u$ such that

$$\begin{aligned} A^T s' \qquad\qquad\qquad &= c \\ -At' - u' &= -b \\ b^T s' - c^T t' \qquad &< 0 \end{aligned}$$

This means that s' is dual feasible and t' is primal feasible, therefore it holds by weak duality that $c^T t' \leq b^T s'$ contradicting $b^T s' < c^T t'$.

2 Strong Duality

Proof. Case 2: $v = 0$

Then s satisfies $s \geq 0$ and $A^T s = 0$, meaning for any $\alpha \geq 0$, $y' + \alpha s$ is dual feasible. Similarly, $-At = u \geq 0$ and therefore, for any $\alpha \geq 0$, $x' + \alpha t$ is primal feasible. By weak duality, this means that, for any $\alpha \geq 0$, it holds that

$$c^T(x' + \alpha t) \leq b^T(y' + \alpha s)$$

$$\iff$$

$$c^T x' - b^T y' \leq \alpha(b^T s - c^T t)$$

The right-hand side tends to $-\infty$ as α tends to ∞, which is a contradiction as the left-hand side is fixed. $\square$

2 Complementary Slackness

Complementary Slackness. If x is a feasible solution to (P) and y is a feasible solution to its dual (D), then x and y are optimal solutions to (P) and (D) respectively, if and only if either $y_i = 0$ or $\sum_j a_{ij}x_j = b_i$ (or both) for all i.

Observation 1. Revisiting the equation in the weak duality proof shows the slack between a feasible and an optimal solution.

$A^T y \geq c$

dual

↓

$$c^T x \leq y^T A x \leq b^T y$$

↑

primal

$Ax \leq b$

2 Complementary Slackness

Observation 2. Given an optimal solution x^* to (P), complementary slackness allows to compute an optimal solution to (D) from x^*, instead of solving the dual using an LP algorithm.

Example.

- Solve (D_*) to obtain an optimal solution $y^* \in \mathbb{R}^2_{\geq 0}$
- Compute an optimal solution $x^* \in \mathbb{R}^4_{\geq 0}$ to (P_*) from $y^* \in \mathbb{R}^2_{\geq 0}$ using complementary slackness

$$(P_*) \quad \begin{aligned} \max \ & (13 \ \ 23 \ \ 2 \ \ 1)^T \cdot x \\ & \begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{pmatrix} \cdot x \leq \begin{pmatrix} 6 \\ 8 \end{pmatrix} \\ & x \in \mathbb{R}^4_{\geq 0} \end{aligned}$$

$$\begin{aligned} \min \ & (6 \ \ 8)^T \cdot y \\ & \begin{pmatrix} 2 & 6 \\ 3 & 7 \\ 4 & 8 \\ 5 & 9 \end{pmatrix} \cdot y \geq \begin{pmatrix} 13 \\ 23 \\ 2 \\ 1 \end{pmatrix} \\ & y \in \mathbb{R}^2_{\geq 0} \end{aligned} \quad (D_*)$$

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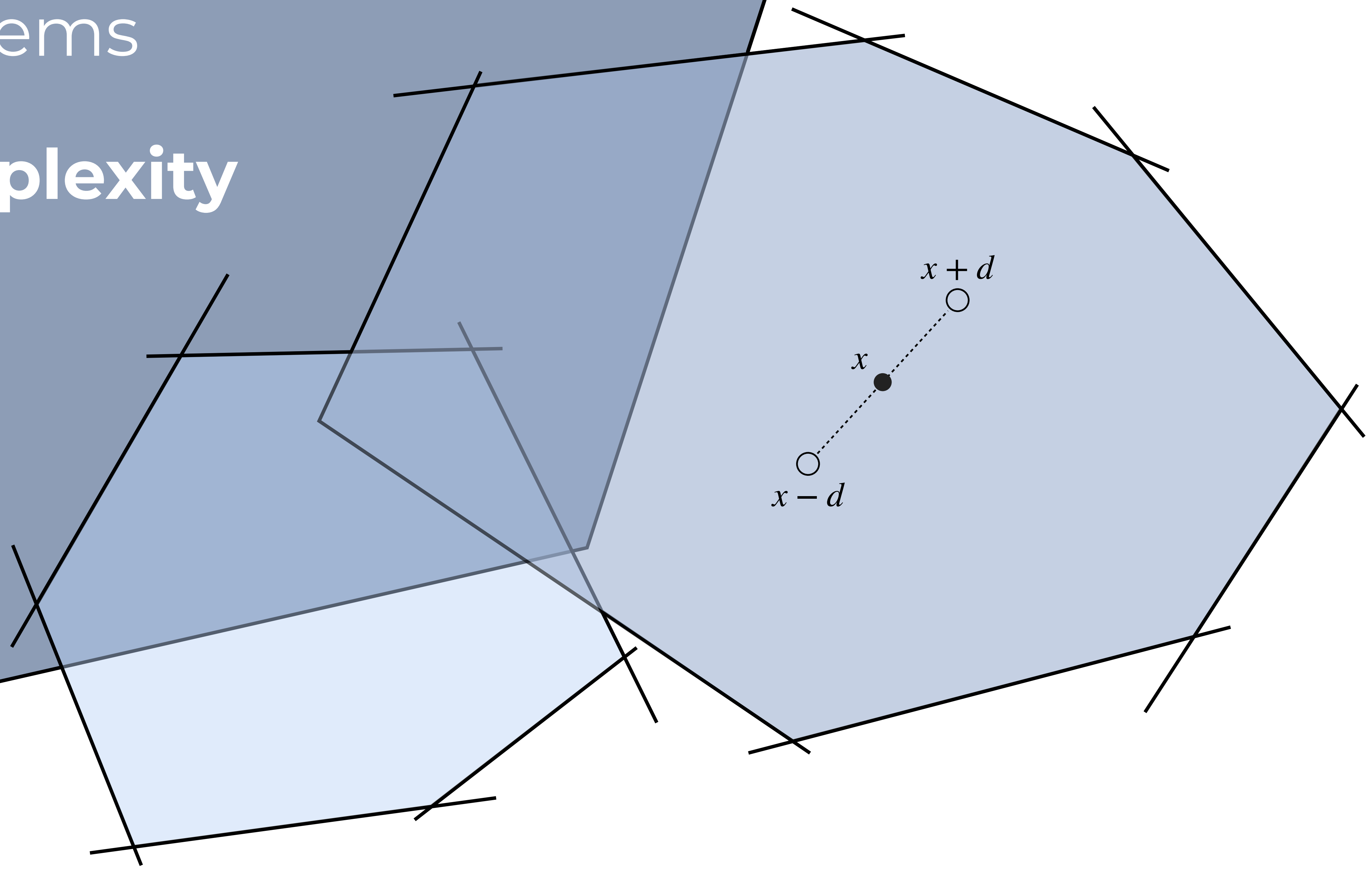
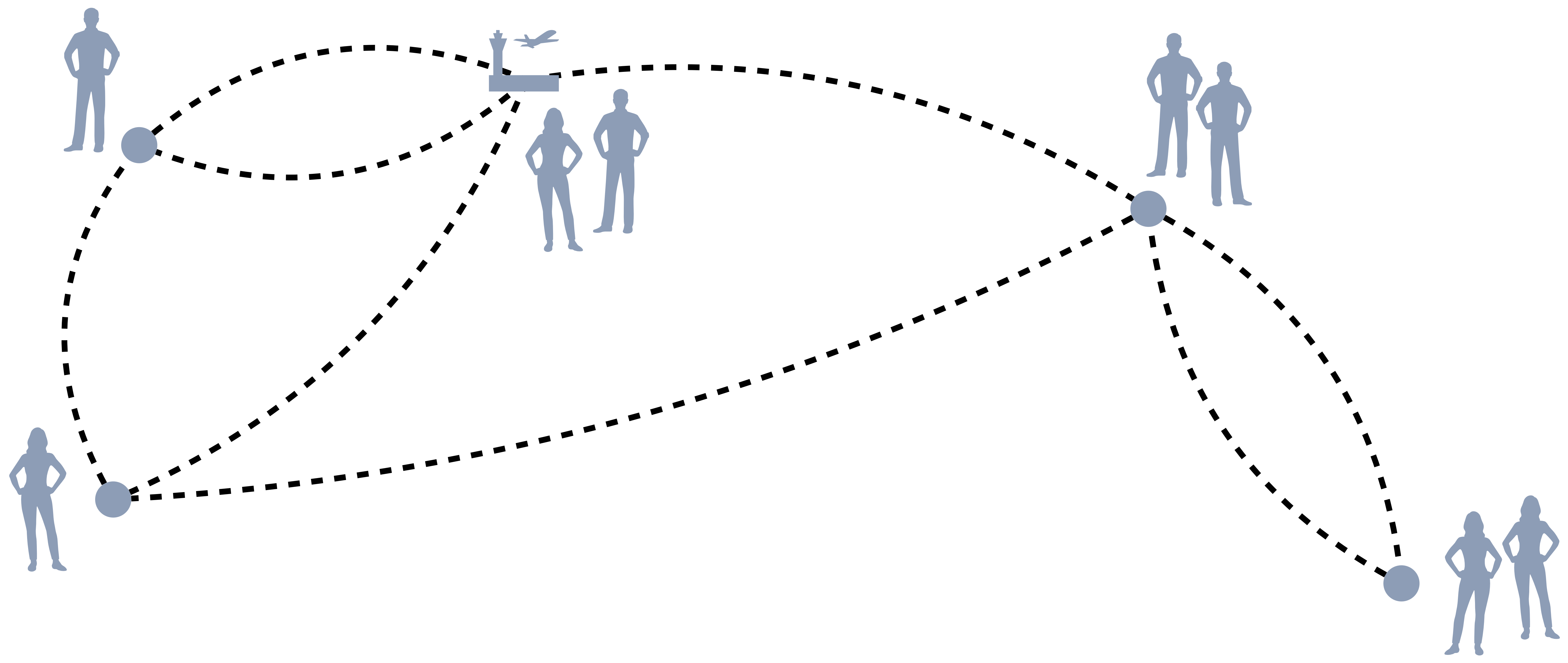


3 Scheduling Example: ILP



$$\max \quad c^T x$$

$$A \in \mathbb{Z}^{m \times n} \left\{ \quad Ax = b \quad \right\} b \in \mathbb{Z}^m$$

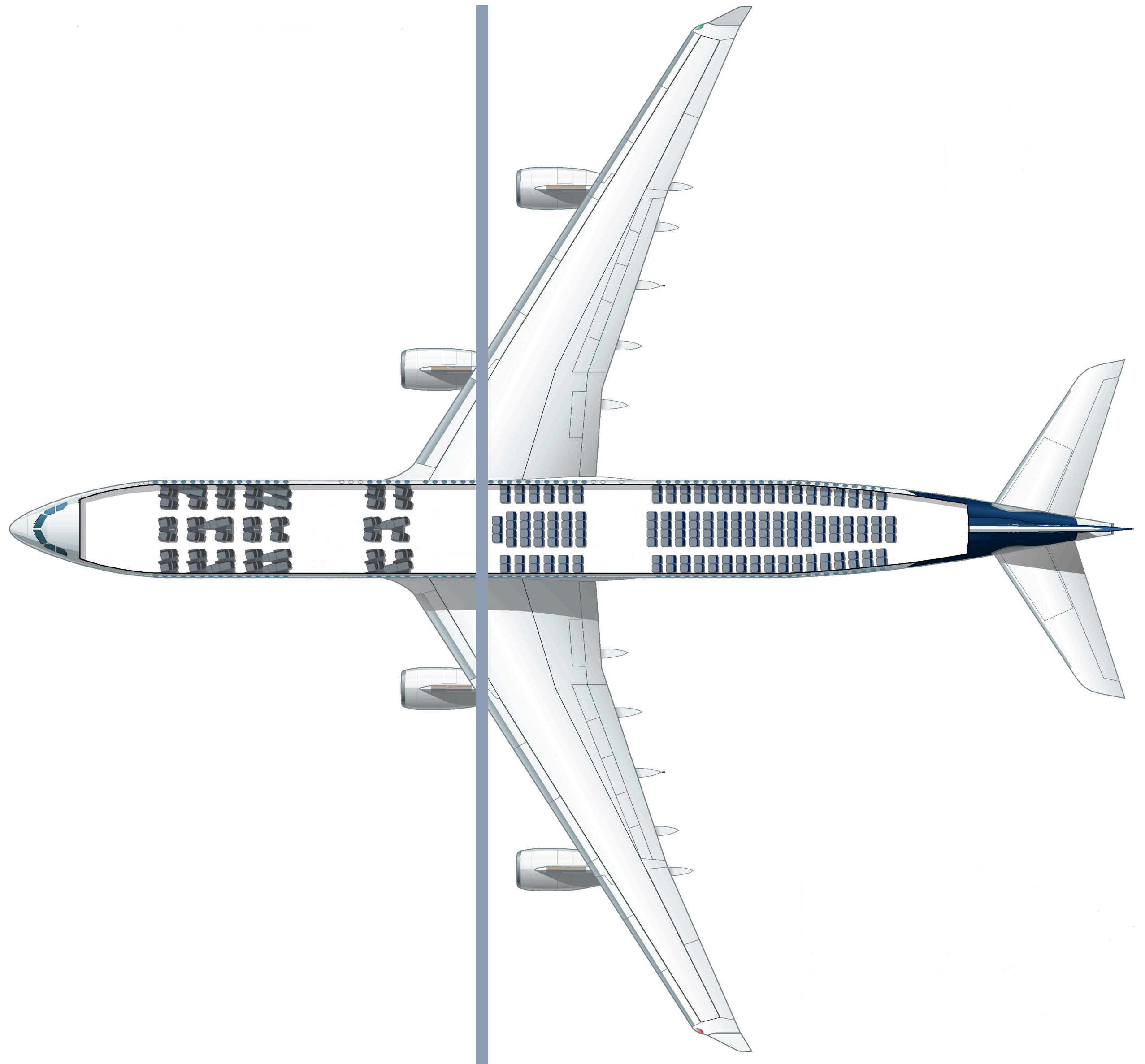
$$x \in \mathbb{Z}^n_{\geq 0}$$

ILP

3 Airline Example: MILP

Business Class

Economy Class



ILP

LP

$$\max \quad c^T x + d^T y$$

$$A \in \mathbb{Z}^{m \times (n_1 + n_2)} \left\{ \begin{pmatrix} A_1 & A_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = b \right\} b \in \mathbb{Z}^m$$

$$x \in \mathbb{Z}^{n_1}_{\geq 0} \qquad y \in \mathbb{R}^{n_2}_{\geq 0}$$

MILP

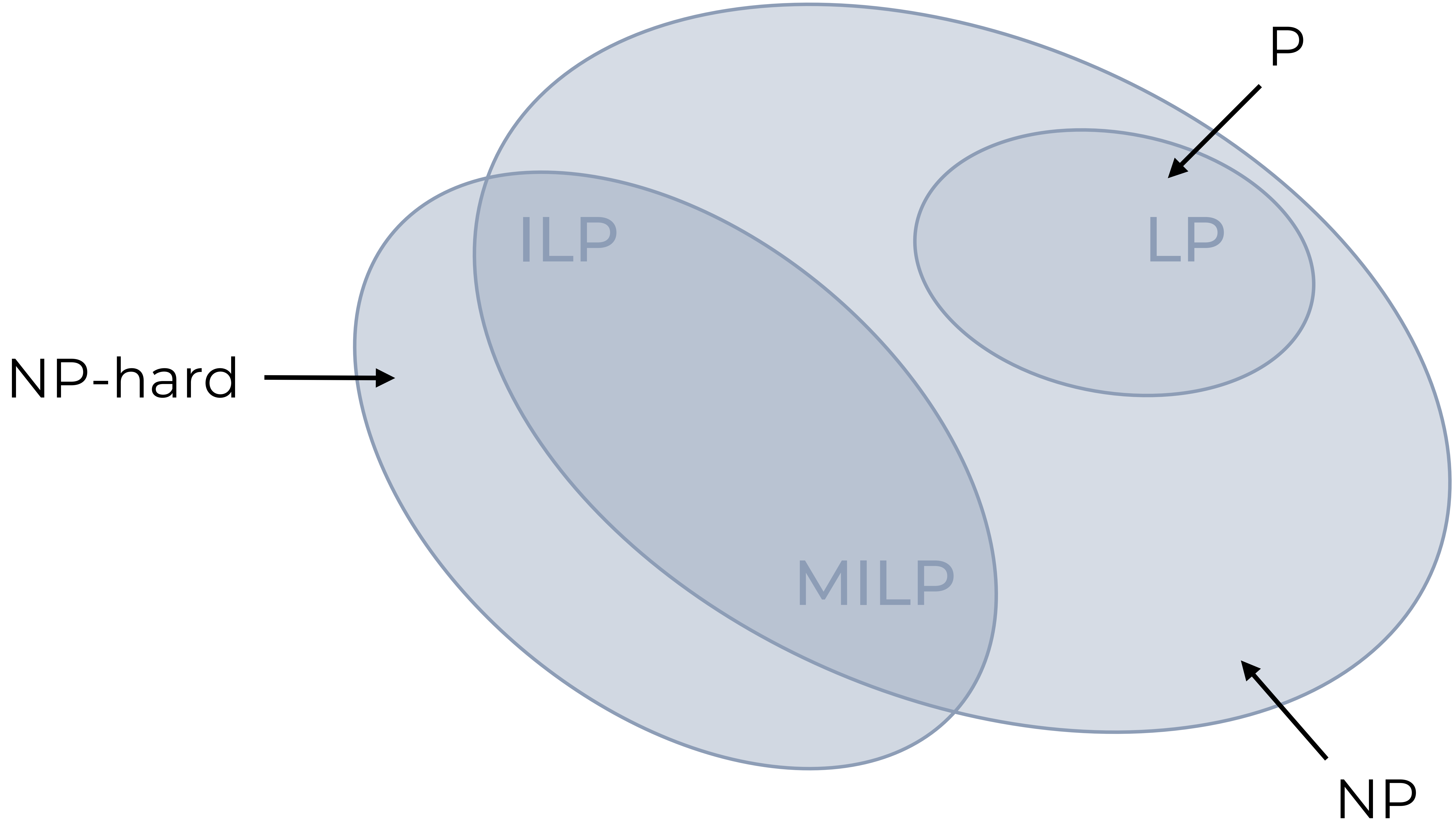
3 Different Program Formulations

$$\left.\begin{aligned} A &\in \mathbb{Z}^{m\times n}, \\ b &\in \mathbb{Z}^{m}, \\ c &\in \mathbb{Z}^{n}, \\ x &\in \mathbb{Z}^{n}_{\geq 0} \end{aligned}\right\} \text{ILP}$$

$$\text{LP} \left\{\begin{aligned} A &\in \mathbb{R}^{m\times n}, \\ b &\in \mathbb{R}^{m}, \\ c &\in \mathbb{R}^{n}, \\ x &\in \mathbb{R}^{n}_{\geq 0} \end{aligned}\right.$$

$$\overbrace{\begin{aligned} & A \in \mathbb{Z}^{m\times (n_1+n_2)} \\ & b \in \mathbb{Z}^{m}, \\ & c \in \mathbb{Z}^{n_1}, d \in \mathbb{Z}^{n_2}, \\ & x \in \mathbb{Z}^{n_1}_{\geq 0}, y \in \mathbb{R}^{n_2}_{\geq 0} \end{aligned}}^{\text{MILP}}$$

3 (M)(I)LP Complexity

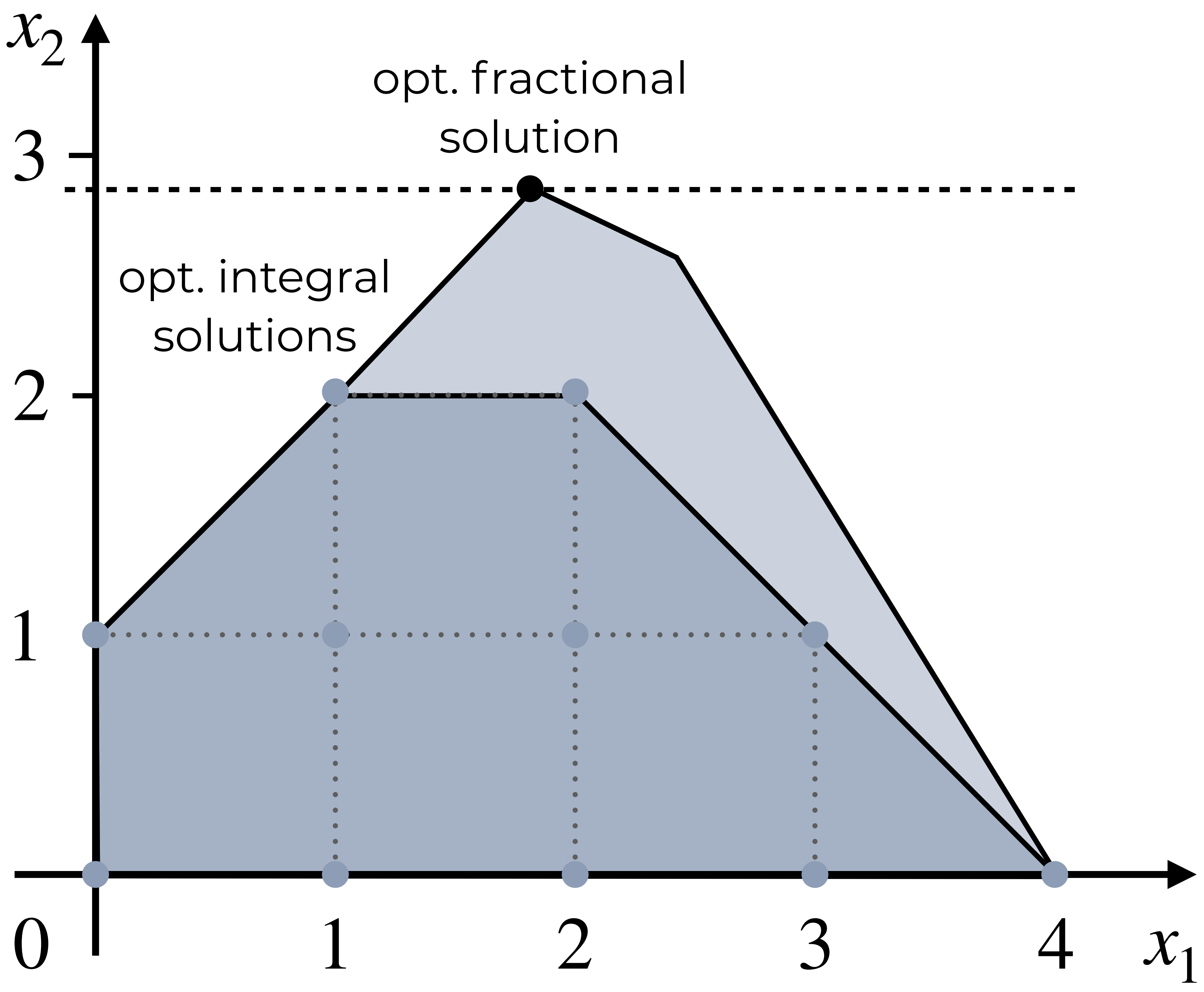


3 LP Relaxation

LP Relaxation. Given a (mixed) integer linear program, the LP which arises by dropping the integrality constraint of each variable is called its LP relaxation.

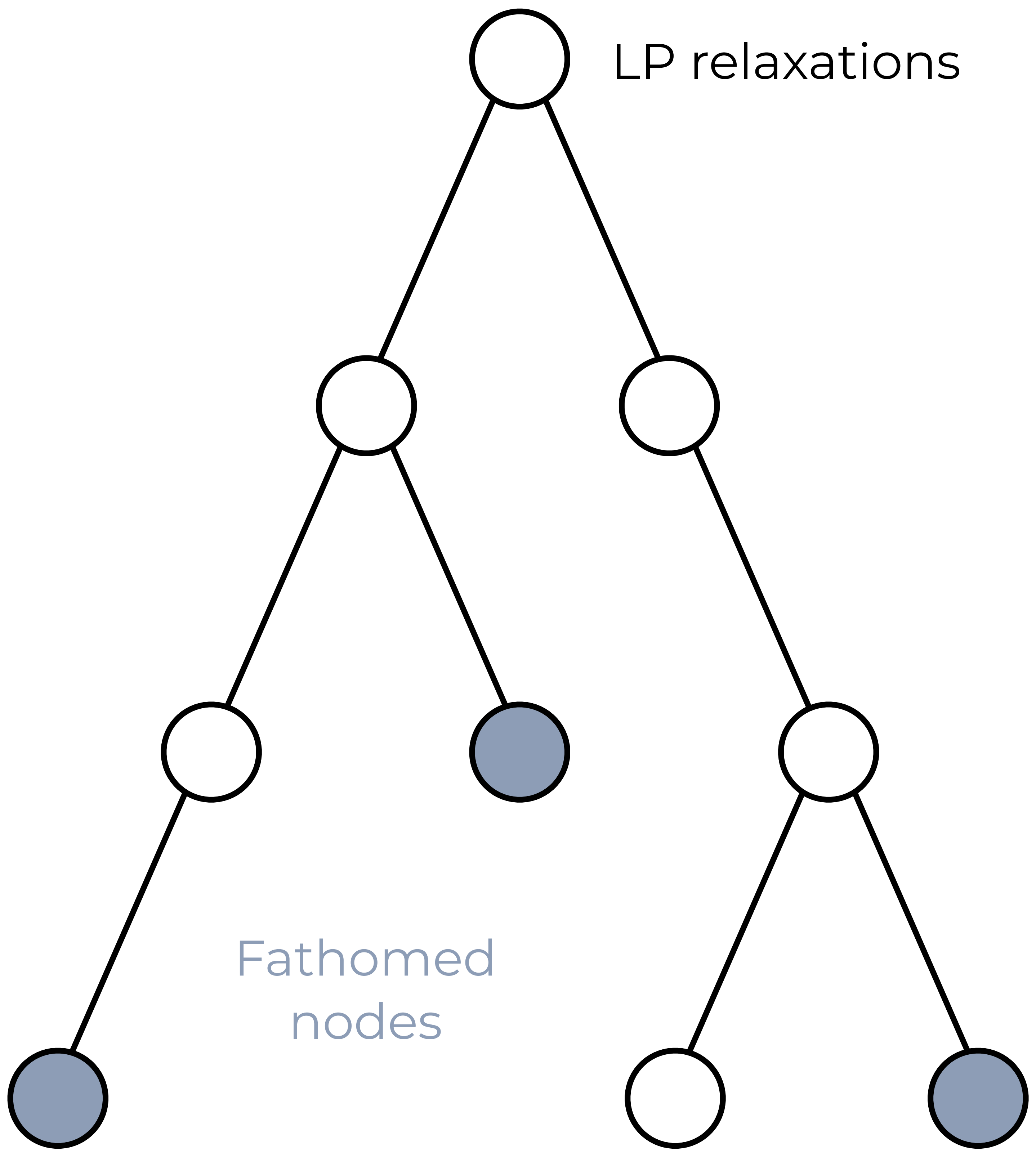
Observation. This technique transforms an NP-hard optimization problem into a related problem solvable in polynomial time.

$$
\begin{array}{rcrl}
\max & & x_2 & \\
-x_1 & + & x_2 & \leq 1 \\
3x_1 & + & 2x_2 & \leq 12 \\
2x_1 & + & 3x_2 & \leq 12 \\
x_1, & & x_2 & \in \mathbb{Z}^2_{\geq 0} \Rightarrow \mathbb{R}^2_{\geq 0}
\end{array}
$$

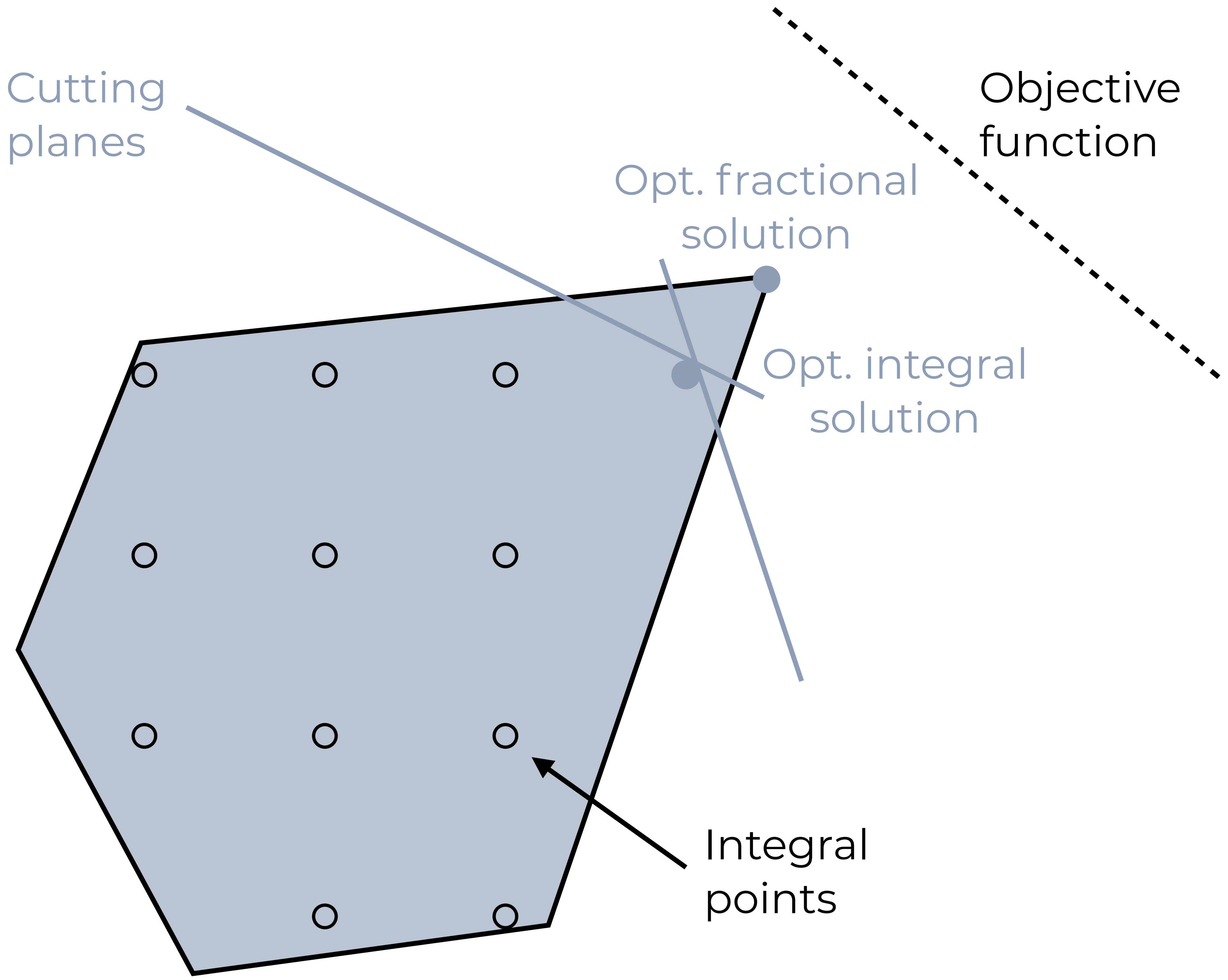


(M)ILP Algorithms

Branch and Bound

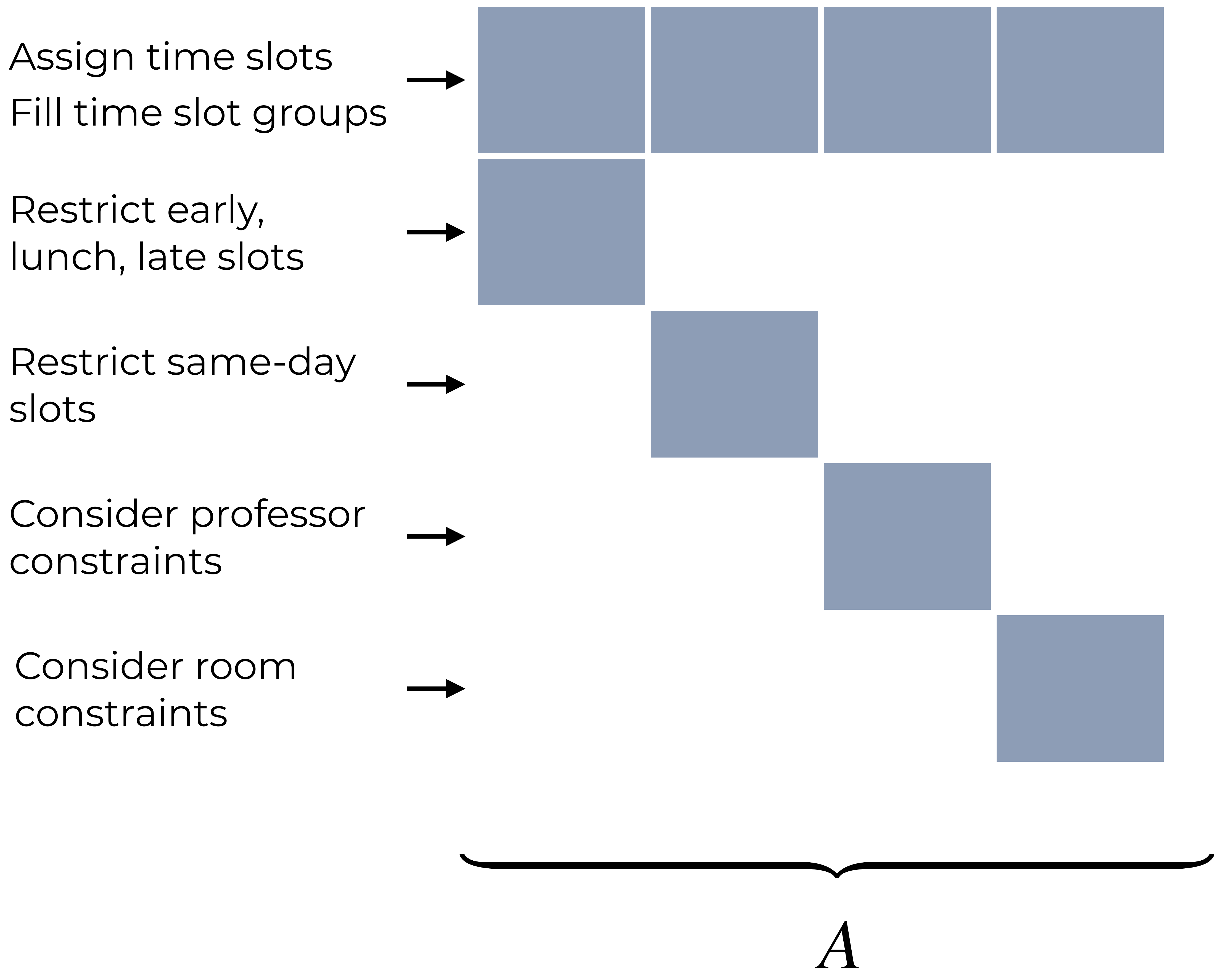


Cutting Plane Method



3 Sparse Constraint Matrices: n-fold

	Mon	Tue	Wed	Thu	Fri
8 - 10					Lecture 1
10 - 12		Lecture 1 Lecture 2			
12 - 14	Lecture 2			Lecture 1 Lecture 3	
14 - 16	Lecture 2 Lecture 3				
16 - 18				Lecture 3	



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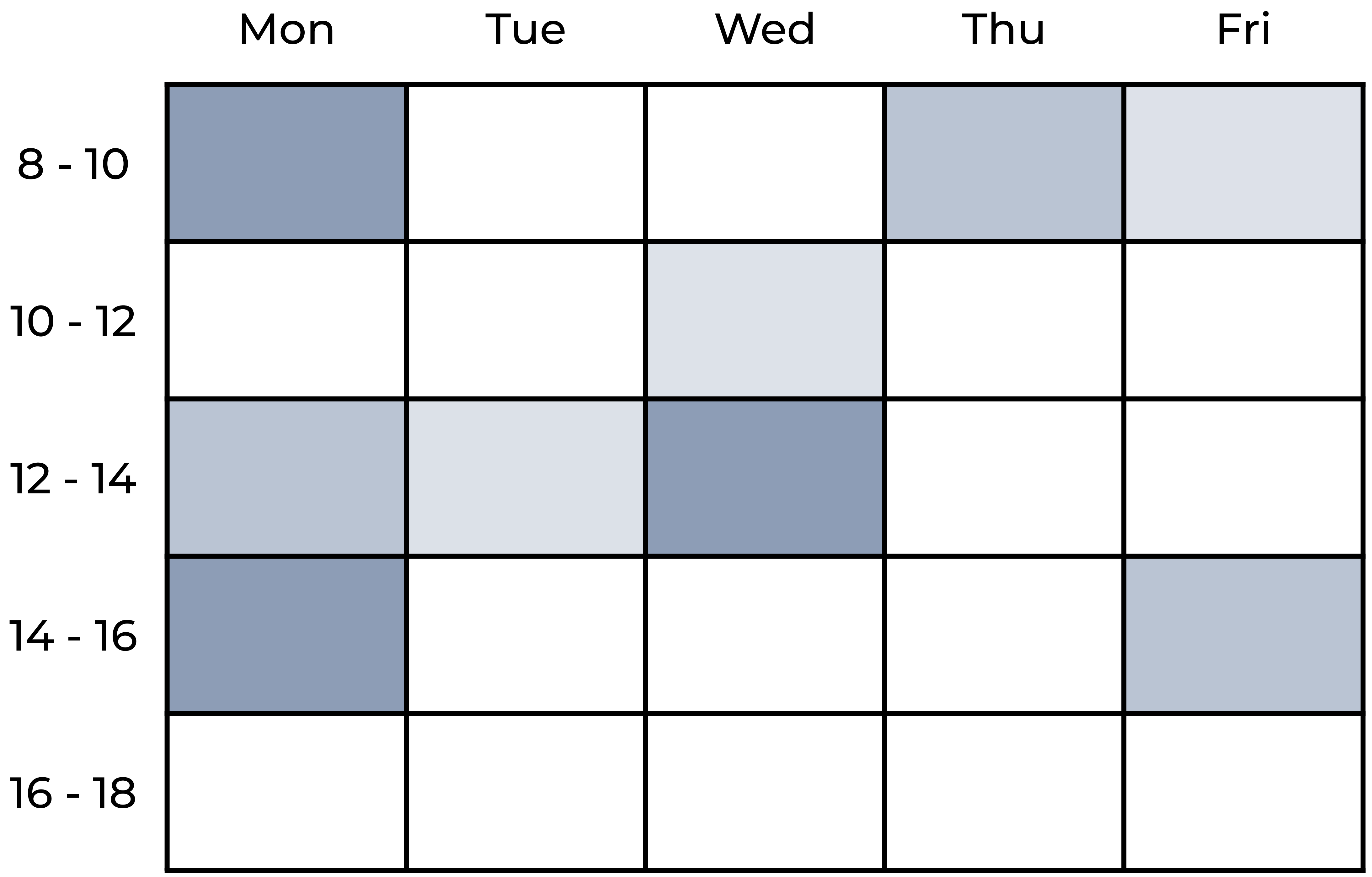
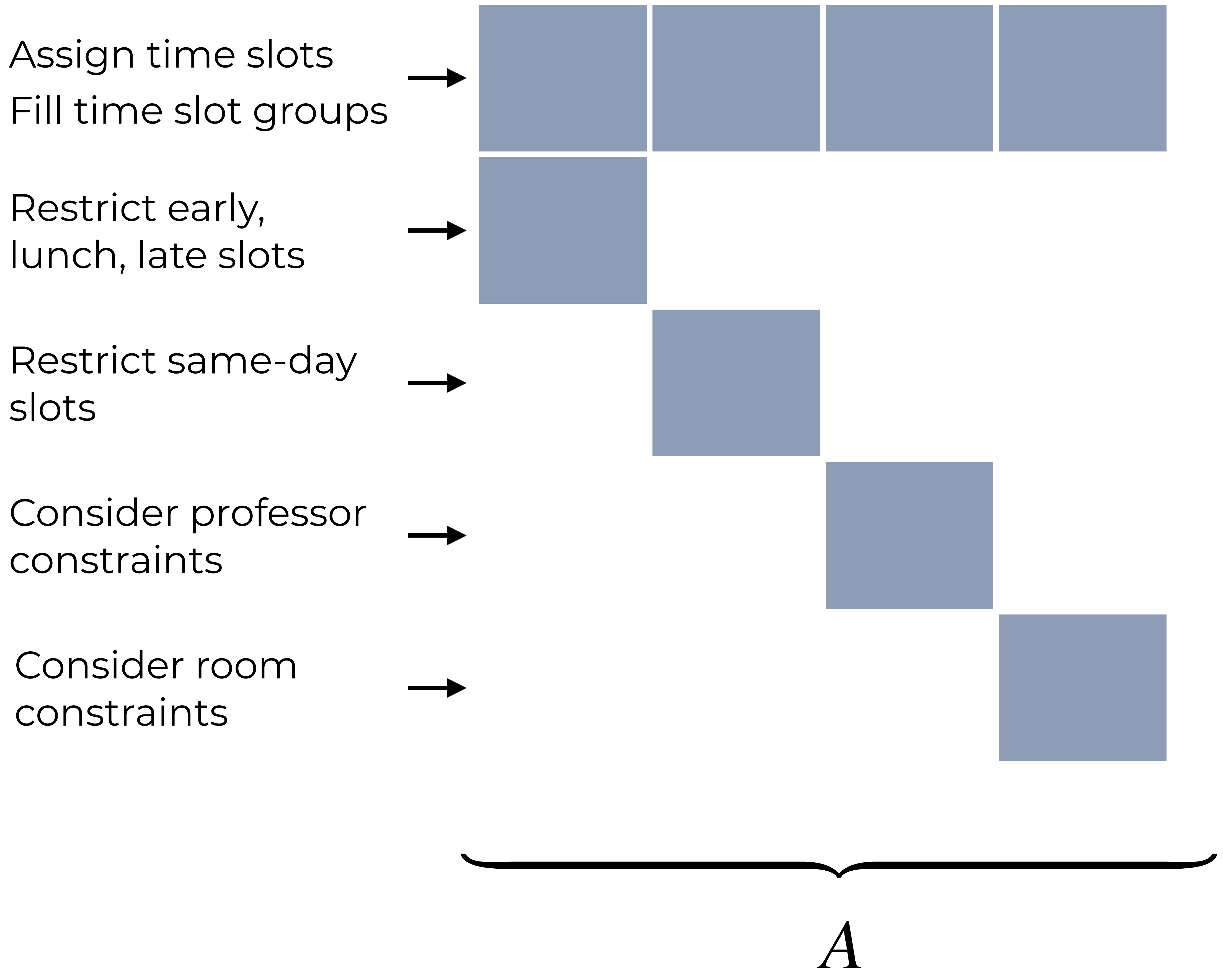
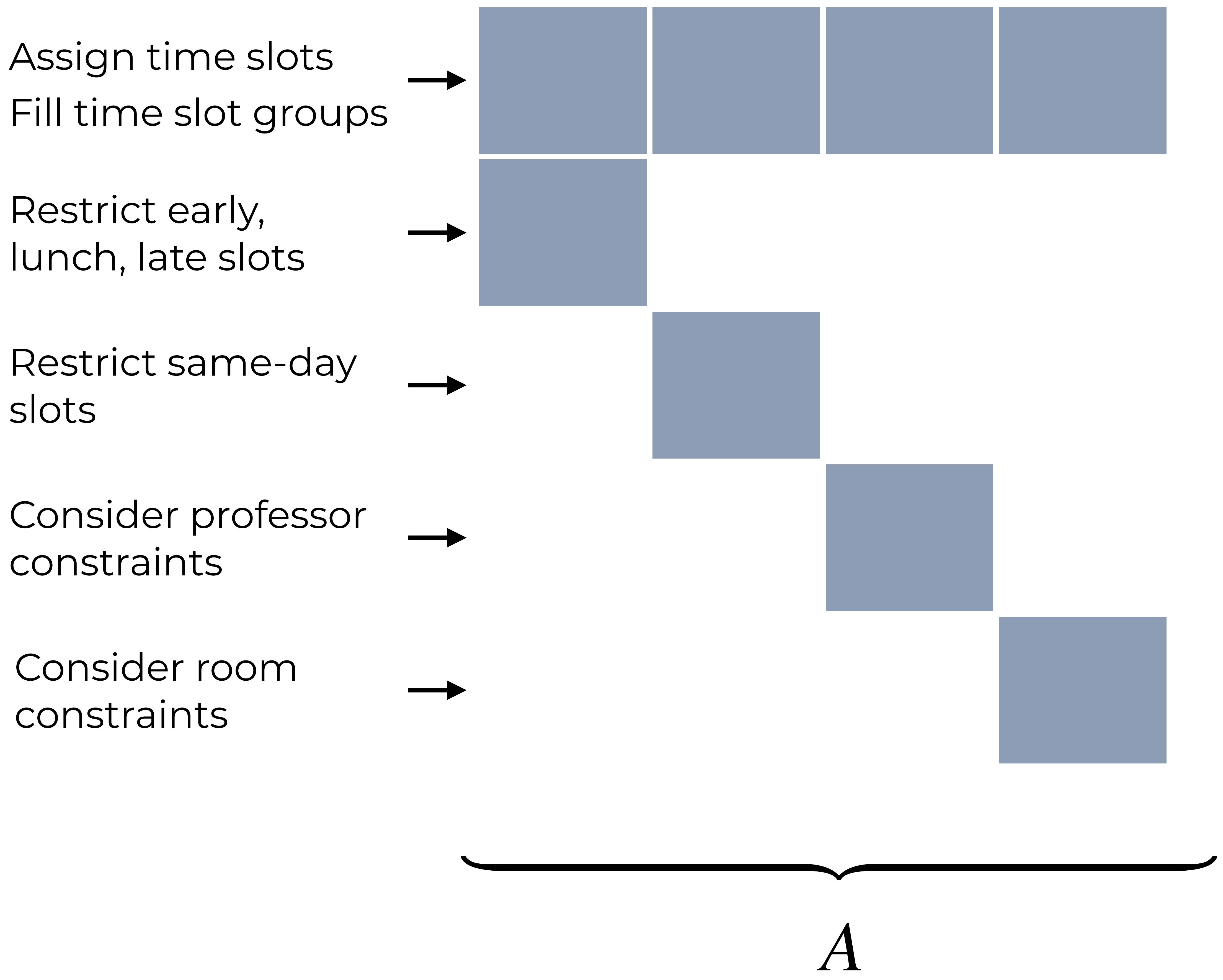




3 Sparse Constraint Matrices: n-fold

	Mon	Tue	Wed	Thu	Fri
8 - 10	Lecture 1			Lecture 2	Lecture 3
10 - 12			Lecture 3		
12 - 14	Lecture 2	Lecture 3	Lecture 1		
14 - 16	Lecture 1				Lecture 2
16 - 18					



3 Sparse Constraint Matrices: two-stage

All-Pairs Shortest Paths

Network Flow

Amortized Analysis

Randomized Algorithms

Hardness

Computability

Linear Programming

Approximation Algorithms

3 Sparse Constraint Matrices: two-stage

- All-Pairs Shortest Paths
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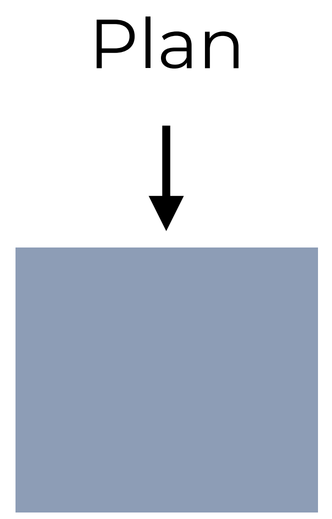

3 Sparse Constraint Matrices: two-stage

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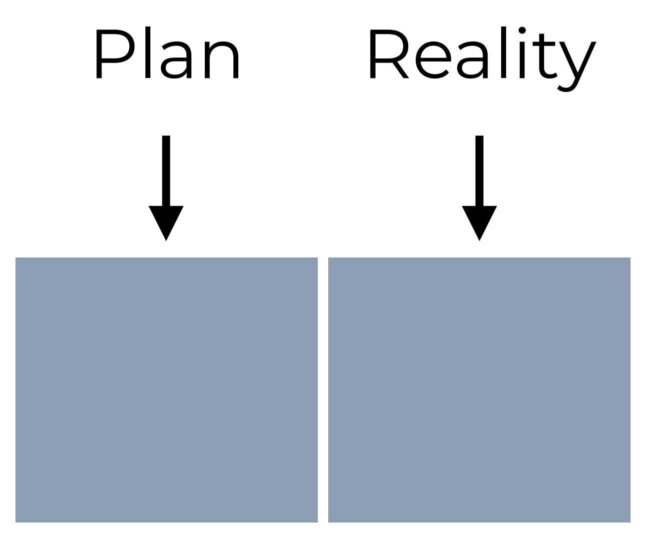

3 Sparse Constraint Matrices: two-stage

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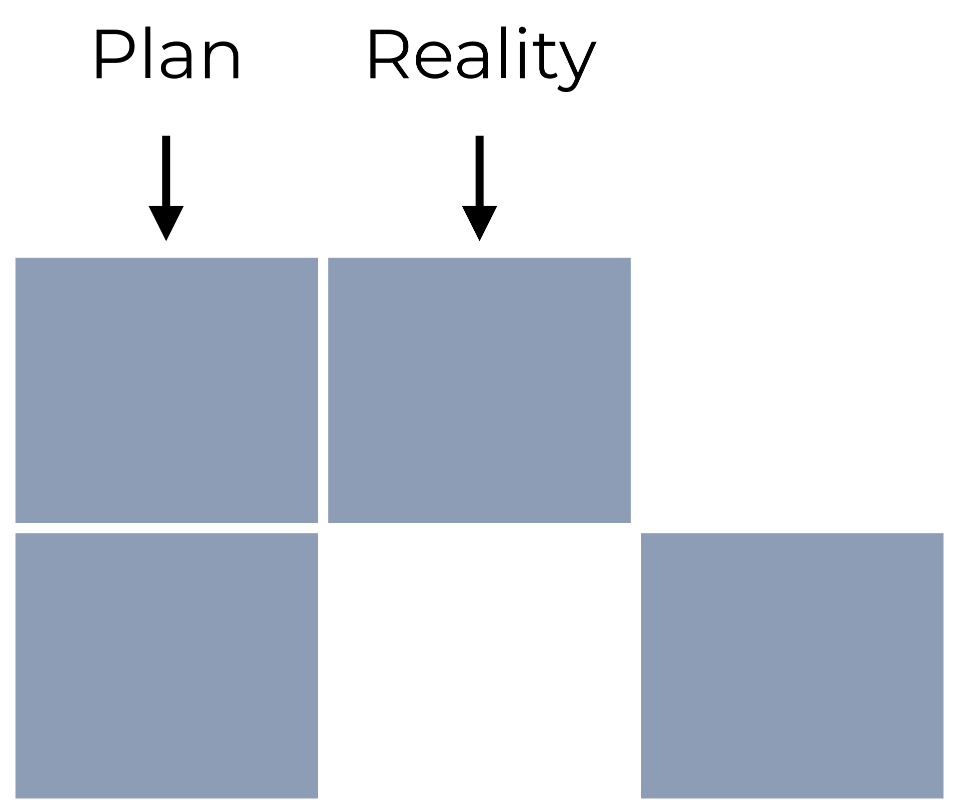
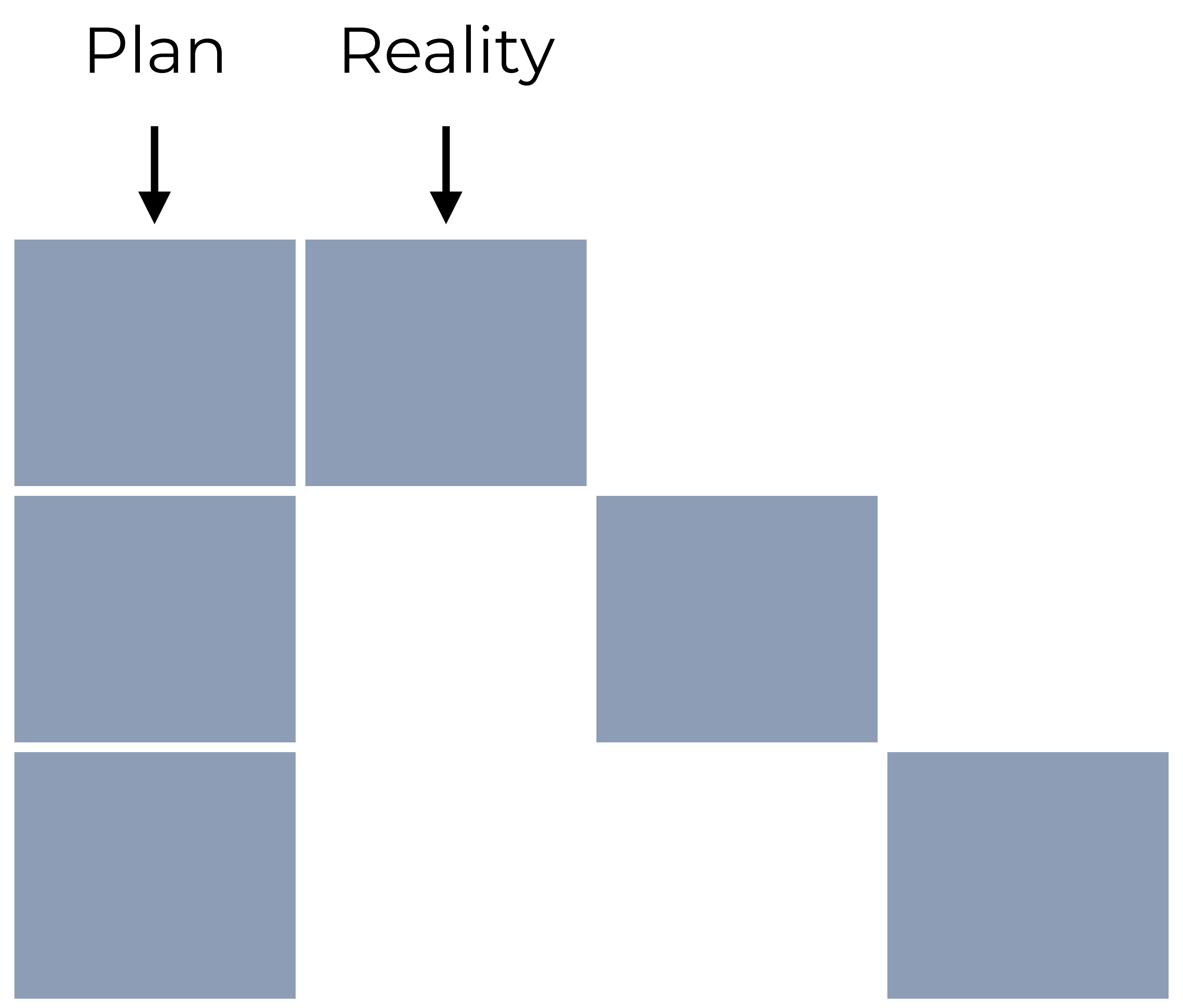
Approximation Algorithms



3 Sparse Constraint Matrices: two-stage

- All-Pairs Shortest Paths
- Network Flow
- Amortized Analysis
- Randomized Algorithms
- Hardness
- Computability
- Linear Programming
- Approximation Algorithms



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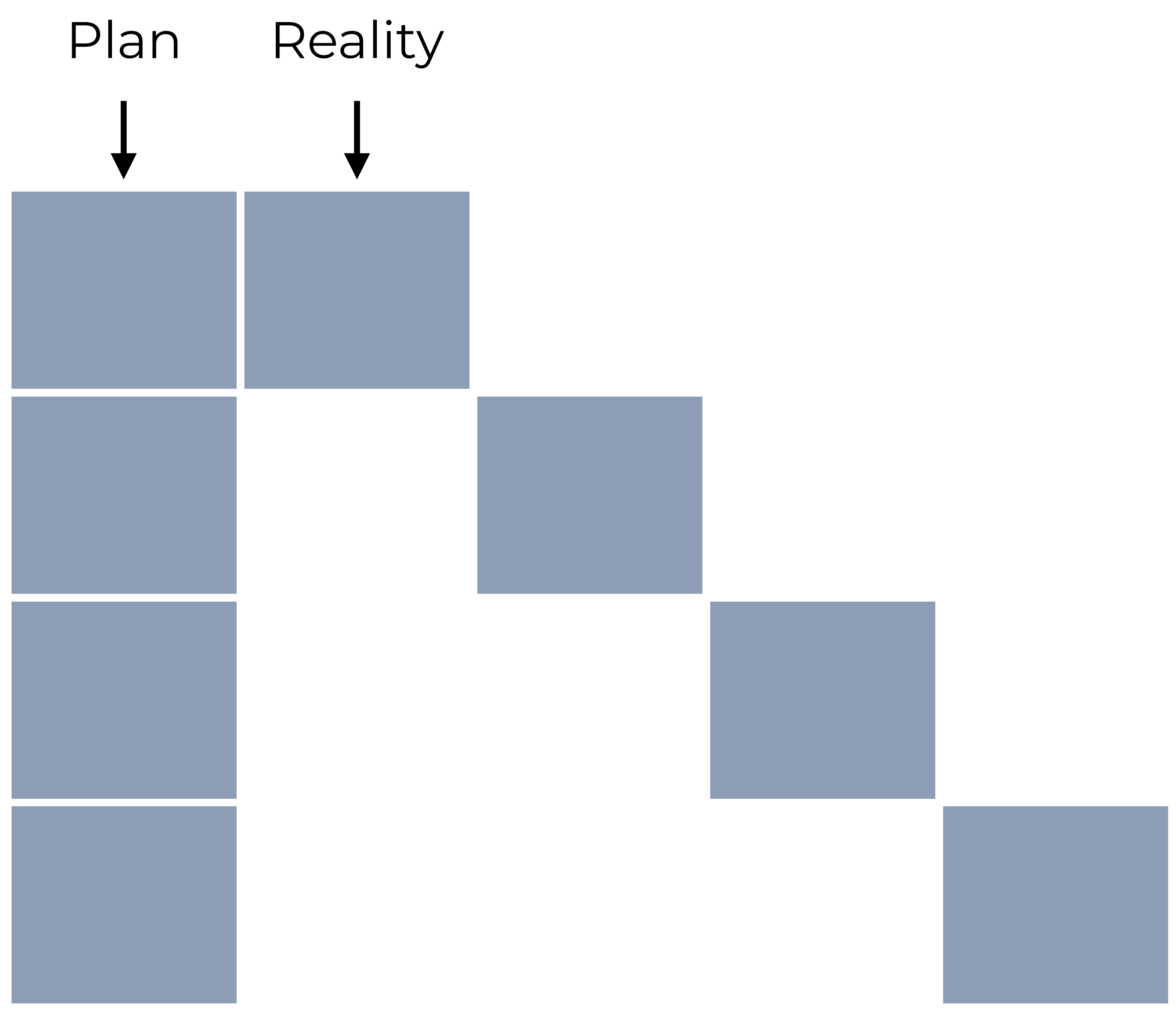

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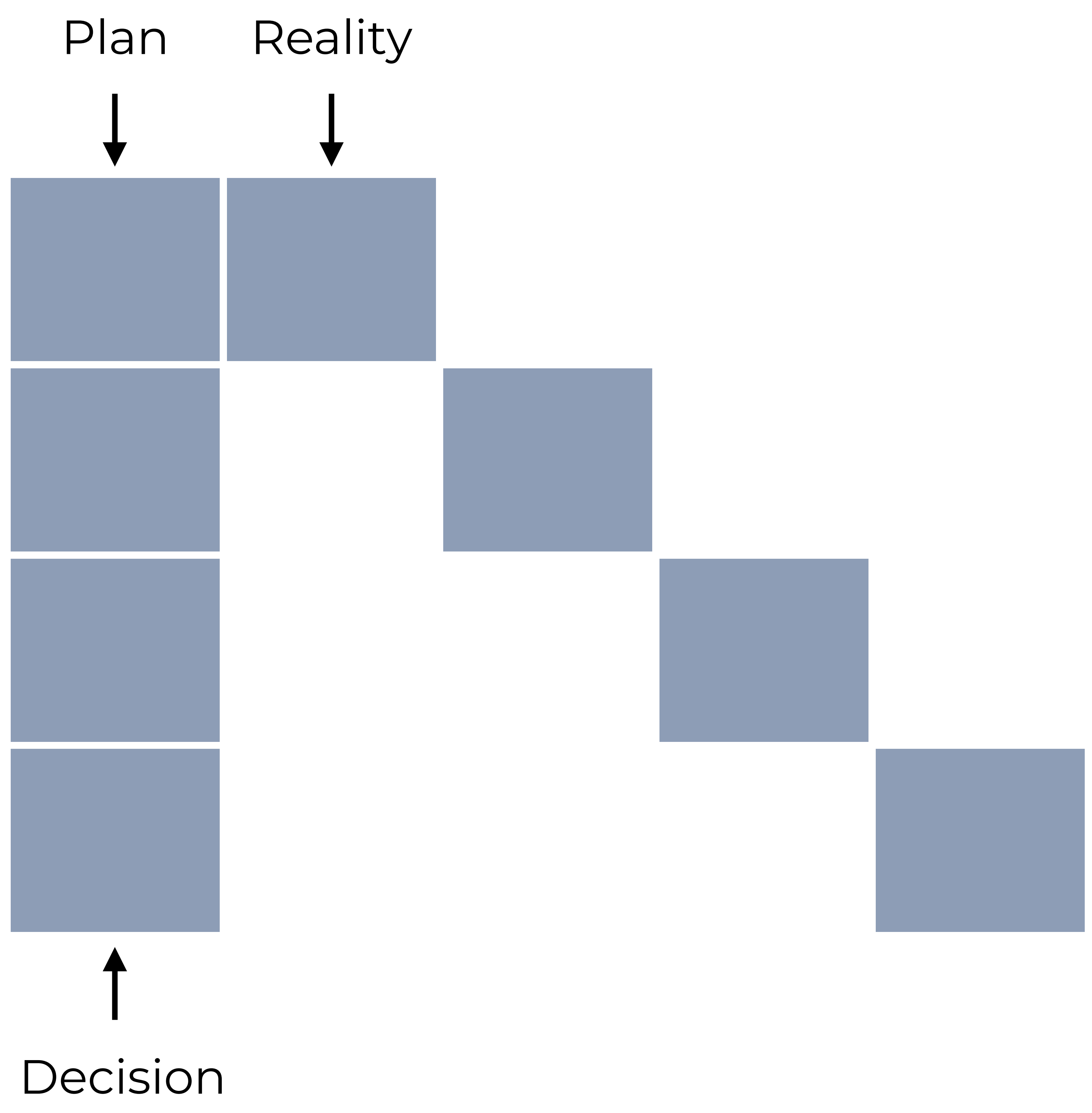

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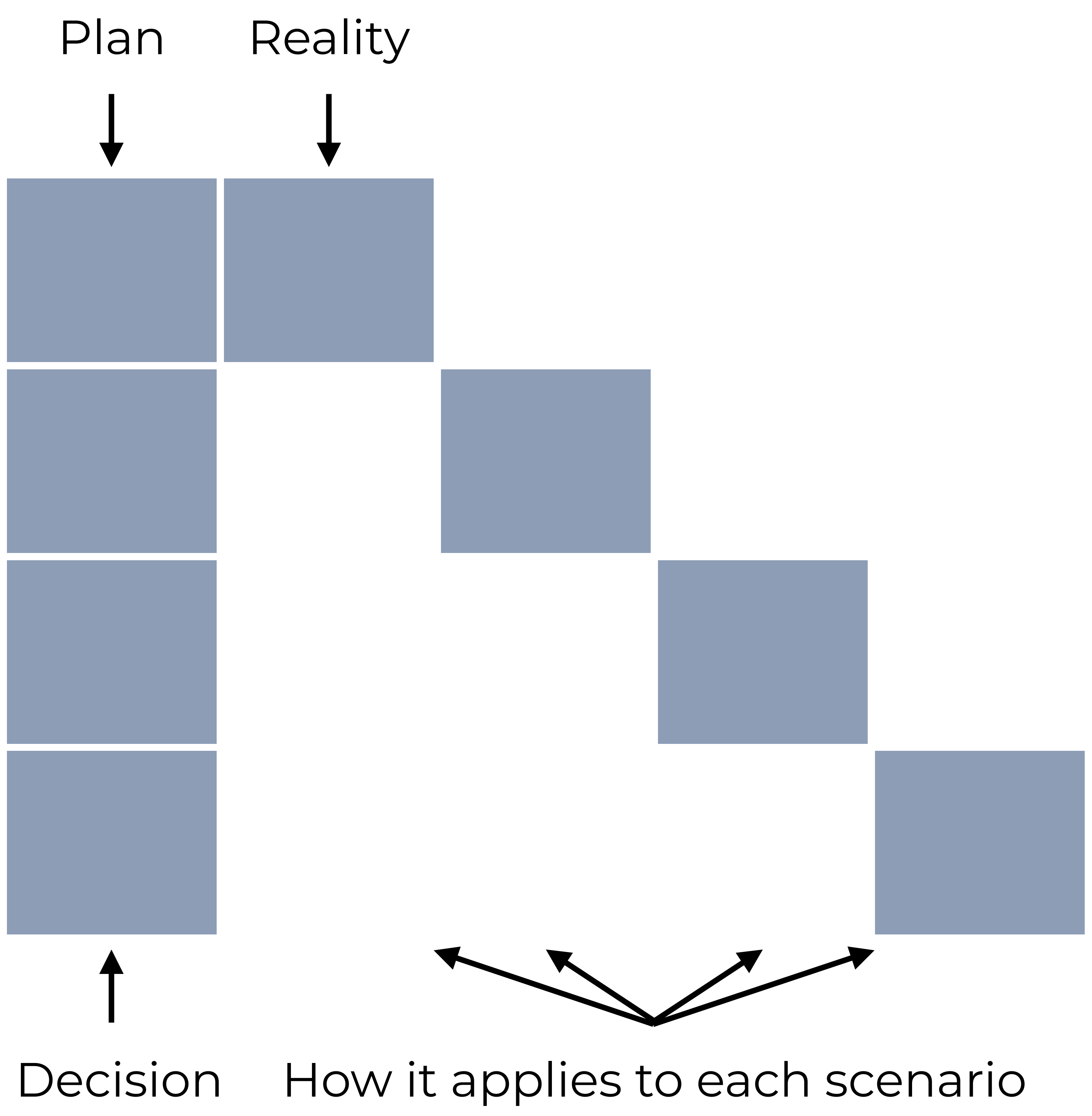

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